Homework 11 for
MATH 497A, Introduction to Ramsey Theory

Due: Monday December 5

Problem 1 – Fundamental sequences

The *Cantor Normal Form* says that every ordinal α can be represented uniquely in the form

$$\alpha = \omega^{\beta_1} k_1 + \cdots + \omega^{\beta_n} k_n$$

where $\alpha \geq \beta_1 > \beta_2 > \cdots > \beta_n$, and $k_1, \dots, k_n$ are nonzero natural numbers.

Note that we may have $\beta_1 = \alpha$, if $\alpha = \omega^\alpha$. However, if $\alpha < \varepsilon_0$, then $\beta_1 < \alpha$.

Use the Cantor Normal Form to construct for every $\alpha < \varepsilon_0$ a fundamental sequence α_n, i.e. an increasing sequence of ordinals such that $\alpha_n \to \alpha$.

Problem 2 – The infinite Pigeonhole Principle

Deduce the infinite Pigeonhole Principle from the recurrence principle for sets, i.e. the fact that for every open covering $(U_i)_{i \in I}$ of $A^{\mathbb{Z}}$, A a finite set, one U_i is recurrent, i.e. $U_i \cap T^n U_i \neq \emptyset$ for infinitely many n.

Problem 3 – Another application of the Hales-Jewett Theorem

Show that if $\mathbb{Z}^d$ is r-colored, and $F \subseteq \mathbb{Z}^d$ is finite, then there exist $v \in \mathbb{Z}^d$ and $r \neq 0$ such that all points of the form

$$v + rw, \qquad w \in F$$

are of the same color.

Problem 4 – Same Theorem, different proof

Prove the statement of #3 again, but this time using the following result.

> Let X be a compact topological space, and let $T_1, \dots, T_n$ be commuting homeomorphisms on X. Let $(U_i)_{i \in I}$ be an open cover of X. Then there exists U_i such that
>
> $$T_1^{-r} U_i \cap \cdots \cap T_n^{-r} U_i \neq \emptyset.$$
>
> for infinitely many r.

Problem 5 – Chaoticness of the shift map

A topological dynamical system (X, T) is commonly called *chaotic* if it satisfies three conditions:

1. *Topological transitivity*. For any two open sets U, V, there exists an $n \in \mathbb{Z}$ such that $T^n(U) \cap V \neq \emptyset$.
2. *The periodic orbits are dense.* The set $\{x : \exists n \in \mathbb{Z}\; T^n x = x\}$ is dense in X.
3. *Sensitivity to initial conditions.* There exists a $\beta > 0$ such that for any $x \in X$ and any $\varepsilon > 0$, there exists a $y \in X$, $k \in \mathbb{Z}^+$ such that $d(x, y) < \epsilon$ but $d(T^n x, T^n y) > \beta$.

Show that the shift map on $2^{\mathbb{Z}}$ has all three properties.