Homework 10 for
MATH 497A, Introduction to Ramsey Theory

Solutions

Problem 1 – Non-standard models of arithmetic, part I

Consider the language $\mathcal{L} = \{S, +, \underline{0}\}$, where S is a unary function symbol, $+$ is a binary function symbol, and $\underline{0}$ is a constant symbol.

Consider the first four Peano axioms:

(P1) $\forall x (S(x) \neq \underline{0})$

(P2) $\forall x \forall y\ (S(x) = S(y) \rightarrow x = y)$

(P3) $\forall x (x + \underline{0} = x)$

(P4) $\forall x \forall y\ x + S(y) = S(x + y)$

A structure satisfying these sentences is $\mathcal{M} = (\mathbb{N}, +1, +, 0)$, i.e. S is interpreted as adding 1, $+$ is interpreted as the usual addition of natural numbers, and $\underline{0}$ is interpreted as the number 0. Find three other (mutually non-isomorphic) structures that satisfy these sentences, but that are not isomorphic to $\mathcal{M}$.

(*Hint:* For example, you could add new elements to $\mathbb{N}$ and interpret the functions on those elements appropriately.)

Solution.

1.) Add an element ω to the standard model, i.e. $M_1 = \mathbb{N} \dot\cup \{\omega\}$ and define

$$S(\omega) = \omega$$
$$\omega + n = n + \omega = \omega + \omega = \omega$$

It is easily verified that the axioms hold in $\mathcal{M}_1 = (M_1, S, +, 0)$. We show that $\mathcal{M}_1$ is not isomorphic to the standard model. Assume there was an isomorphism $h : \mathbb{N} \to \mathcal{M}_1$. Since h is a bijection there exists exactly one $n \in \mathbb{N}$ such that $h(n) = \omega$. This implies $\omega = S(h(n)) \neq h(S(n)) \in \mathbb{N}$, contradiction.

2.) Add a copy $\hat{\mathbb{N}} = \{\hat{n} :\ n \in \mathbb{N}\}$ to $\mathbb{N}$, i.e. $M_2 = \mathbb{N} \dot\cup \hat{\mathbb{N}}$ and define

$$S(\hat{n}) = \widehat{n+1}$$
$$\hat{n} + m = m + \hat{n} = \widehat{n+m}$$
$$\hat{n} + \hat{m} = \widehat{n+m}$$

Again the axioms are easily verified. $\mathcal{M}_2 = (M_2, S, +, 0)$ is not isomorphic to the standard model similar to $\mathcal{M}_1$. Furthermore, $\mathcal{M}_2$ is not isomorphic to $\mathcal{M}_1$: Assume $h : \mathcal{M}_1 \to \mathcal{M}_2$ were an isomorphism. Then there exists $n \in \mathbb{N}$ such that $h(n) = \hat{m} \in \hat{\mathbb{N}}$ but $h(S(n)) \in \mathbb{N}$ (since otherwise only finitely many $n \in \mathbb{N}$ would be mapped to $\mathbb{N}$). It follows that $\mathbb{N} \ni h(S(n)) = S(h(n)) = S(\hat{m}) = \widehat{S(m)} \in \mathbb{N}$, contradiction.

3.) Add the real numbers to the standard model, i.e. $M_3 = \mathbb{N} \dot\cup \mathbb{R}$. Define for $\alpha, \beta \in \mathbb{R}$

$$S(\alpha) = \alpha +^{\mathbb{R}} 1$$
$$\alpha + m = m + \alpha = \alpha +^{\mathbb{R}} m$$
$$\alpha + \beta = \alpha +^{\mathbb{R}} \beta$$

This structure cannot be isomorphic to the other ones since it is uncountable.

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Problem 2 – Models of PA

Show that $\mathbb{R}^{\geq 0} = (\mathbb{R}^{\geq 0}, +^{\mathbb{R}}, \cdot^{\mathbb{R}}, +1, 0)$ is not a model of PA.

Solution. We define

$$x \leq y :\Leftrightarrow \exists z (x + z = y), \quad \text{and} \quad x < y :\Leftrightarrow x \leq y \ \&\ x \neq y.$$

Let $\underline{1} = S(\underline{0})$. By (P1), we have $\underline{0} \neq \underline{1}$.

Using (P4), one can show that

$$\underline{1} \leq y \quad \Leftrightarrow \quad \exists z(S(z) = y).$$

Now use (PInd) to show that

$$\mathsf{PA} \models \forall y[y = \underline{0} \vee [\underline{0} < y \Rightarrow \exists z(S(z) = y)]].$$

Hence

$$\mathsf{PA} \models \forall y\,(\underline{0} < y \Rightarrow \underline{1} \leq y),$$

but $\mathbb{R}^{\geq 0}$ does not satisfy this sentence. ■

Problem 3 – Axiomatization of groups

Let $\mathcal{L} = \{\cdot, \underline{e}\}$ be the language of groups. Find finitely many $\mathcal{L}$-sentences $\Phi = \{\varphi_1, \dots, \varphi_n\}$ such that every model of $\mathsf{GT} \cup \Phi$ is isomorphic to $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

Do the same for $\mathbb{Z}_4$.

Bonus: Is this possible for any distinct finite group? That is, if G is a finite group, does there exist a (finite) set of sentences Φ_G such that every model of $\mathsf{GT} \cup \Phi_G$ is isomorphic to G?

Solution. For $\mathbb{Z}_2$: There is (up to isomorphism) only one group with two elements. Hence it suffices to ensure that any model of $\mathsf{GT} \cup \Phi$ has exactly two elements. This can be done using the sentence

$$\exists x_1, x_2[x_1 \neq x_2 \;\&\; \forall y\,(y = x_1 \vee y = x_2)].$$

For $\mathbb{Z}_4$: There are (up to isomorphism) two groups of order four - $\mathbb{Z}_4$ and the Klein-Four Group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The latter group is not cyclic. So we let, similar to the case $\mathbb{Z}_2$, φ_1 be the statement that there exist exactly 4 elements, and let φ_2 be the sentence

$$\exists x \forall y[y = x \vee y = x \cdot x \vee y = x \cdot x \cdot x \vee y = x \cdot x \cdot x \cdot x].$$

Then $\mathsf{GT} \cup \{\varphi_1, \varphi_2\}$ has only $\mathbb{Z}_4$ as a model.

G an arbitrary finite group of order n: Every finite group is completely determined (up to isomorphism) through its multiplication table. Hence we only need one sentence to define G up to isomorphism: The sentence

$$\exists x_1, \dots x_n\,[\bigwedge_{i \neq j} x_i \neq x_j \quad \& \quad (\textit{relations of the multiplication table of the form } x_i \cdot x_j = x_k)].$$

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Problem 4 – The compactness theorem, again

Fix a language $\mathcal{L}$. Show that a set T of $\mathcal{L}$-sentences has a model if and only if every finite subset of T has a model.

Solution. Clearly, if T has a model then every finite subset of T has a model.

Now assume every finite subset of T has a model. Suppose for a contradiction T does not have a model. Then every model of T is trivially (since there is none) also a model of $\varphi \;\&\; \neg\varphi$ for any sentence φ, that is, $T \models (\varphi \;\&\; \neg\varphi)$. By the completeness theorem, $T \vdash (\varphi \;\&\; \neg\varphi)$, i.e. T is inconsistent. There must exist a finite proof of $(\varphi \;\&\; \neg\varphi)$. This proof can use only finitely many formulas from T. Collect these finitely many formulas in a finite subset $T_0 \subseteq T$. Then $T_0 \vdash (\varphi \;\&\; \neg\varphi)$ and hence $T_0 \models (\varphi \;\&\; \neg\varphi)$. By assumption, T_0 has a model, say $\mathcal{M}$, and $T_0 \models (\varphi \;\&\; \neg\varphi)$ implies that $\mathcal{M} \models (\varphi \;\&\; \neg\varphi)$, which is impossible. ■

Problem 5 – Non-standard models of arithmetic, part II

Let $\mathcal{L} = \{S, +, \cdot, \underline{0}\}$, and let $\mathbb{N}$ be the standard $\mathcal{L}$-structure of the natural numbers.

Let $T_{\mathbb{N}} = \{\varphi : \mathbb{N} \models \varphi\}$. $T_{\mathbb{N}}$ is called the (first-order) *theory of arithmetic*. Use the compactness theorem (above, #3) to show that there exists a model of $T_{\mathbb{N}}$ that is not isomorphic to $\mathbb{N}$.

Solution. Extend the language of arithmetic by adding a new constant symbol $\underline{c}$.

For $n \in \mathbb{N}$, let φ_n be the sentence $\underline{n} < \underline{c}$. Put $T' = T_{\mathbb{N}} \cup \{\varphi_n : n \in \mathbb{N}\}$.

Every finite subset of T' has a model – the standard model $\mathbb{N}$ (we just have to interpret $\underline{c}$ large enough). By the compactness theorem in #4, we infer that T' has a model $\mathcal{M}$. By construction, $\mathcal{M}$ is also a model of $T_{\mathbb{N}}$, But in $\mathcal{M}$ it must hold that c (the interpretation of the constant symbol $\underline{c}$) is greater than every natural number. ■