Homework 10 for
MATH 497A, Introduction to Ramsey Theory

Solutions

Problem 1 – Non-standard models of arithmetic, part I

Consider the language \( \mathcal{L} = \{ S, +, 0 \} \), where \( S \) is a unary function symbol, + is a binary function symbol, and 0 is a constant symbol.

Consider the first four Peano axioms:

(P1) \( \forall x (S(x) \neq 0) \)
(P2) \( \forall x \forall y (S(x) = S(y) \rightarrow x = y) \)
(P3) \( \forall x (x + 0 = x) \)
(P4) \( \forall x \forall y (x + S(y) = S(x + y)) \)

A structure satisfying these sentences is \( M = (\mathbb{N}, +, 0) \), i.e. \( S \) is interpreted as adding 1, + is interpreted as the usual addition of natural numbers, and 0 is interpreted as the number 0. Find three other (mutually non-isomorphic) structures that satisfy these sentences, but that are not isomorphic to \( M \).

(Hint: For example, you could add new elements to \( \mathbb{N} \) and interpret the functions on those elements appropriately.)

Solution.

1.) Add an element \( \omega \) to the standard model, i.e. \( M_1 = \mathbb{N} \cup \{ \omega \} \) and define

\[
S(\omega) = \omega \\
\omega + n = n + \omega = \omega + \omega = \omega
\]

It is easily verified that the axioms hold in \( M_1 = (M_1, S, +, 0) \). We show that \( M_1 \) is not isomorphic to the standard model. Assume there was an isomorphism \( h : \mathbb{N} \rightarrow M_1 \). Since \( h \) is a bijection there exists exactly one \( n \in \mathbb{N} \) such that \( h(n) = \omega \). This implies \( \omega = S(h(n)) \neq h(S(n)) \in \mathbb{N} \), contradiction.

2.) Add a copy \( \hat{\mathbb{N}} = \{ \hat{n} : n \in \mathbb{N} \} \) to \( \mathbb{N} \), i.e. \( M_2 = \mathbb{N} \cup \hat{\mathbb{N}} \) and define

\[
S(n) = n + 1 \\
\hat{n} + m = m + \hat{n} = \hat{n} + m \\
\hat{h} + \hat{m} = \hat{h} + \hat{m}
\]

Again the axioms are easily verified. \( M_2 = (M_2, S, +, 0) \) is not isomorphic to the standard model similar to \( M_1 \). Furthermore, \( M_2 \) is not isomorphic to \( M_1 \): Assume \( h : M_1 \rightarrow M_2 \) were an isomorphism. Then there exists \( n \in \mathbb{N} \) such that \( h(n) = \hat{m} \in \hat{\mathbb{N}} \) but \( h(S(n)) \in \mathbb{N} \) (since otherwise only finitely many \( n \in \mathbb{N} \) would be mapped to \( \hat{\mathbb{N}} \)). It follows that \( \exists n \in \mathbb{N} : h(S(n)) = S(h(n)) = S(\hat{m}) = S(\hat{n}) \in \mathbb{N} \), contradiction.

3.) Add the real numbers to the standard model, i.e. \( M_3 = \mathbb{N} \cup \hat{\mathbb{R}} \). Define for \( \alpha, \beta \in \mathbb{R} \)

\[
S(\alpha) = \alpha + 1 \\
\alpha + m = m + \alpha = \alpha + R m \\
\alpha + \beta = \alpha + R \beta
\]

This structure cannot be isomorphic to the other ones since it is uncountable.

Problem 2 – Models of PA

Show that \( \mathbb{R}^{\geq 0} = (\mathbb{R}^{\geq 0}, +, R, 1, 0) \) is not a model of PA.

Solution. We define

\[ x \leq y \iff \exists z (x + z = y), \quad \text{and} \quad x < y \iff x \leq y \land x \neq y. \]

Let \( \mathbf{1} = S(0) \). By (P1), we have \( 0 \neq \mathbf{1} \).
Using (P4), one can show that
\[ 1 \leq y \iff \exists z (S(z) = y). \]
Now use (PInd) to show that
\[ \text{PA} \models \forall y (y = 0 \lor [0 < y \Rightarrow \exists z (S(z) = y)]). \]
Hence
\[ \text{PA} \models \forall y (0 < y \Rightarrow 1 \leq y), \]
but \( \mathbb{R}^{\geq 0} \) does not satisfy this sentence.

\section*{Problem 3 – Axiomatization of groups}

Let \( \mathcal{L} = \{ \cdot, 0 \} \) be the language of groups. Find finitely many \( \mathcal{L} \)-sentences \( \Phi = \{ \varphi_1, \ldots, \varphi_n \} \) such that every model of \( \text{GT} \cup \Phi \) is isomorphic to \( \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \).

Do the same for \( \mathbb{Z}_4 \).

\textbf{Bonus:} Is this possible for any distinct finite group? That is, if \( G \) is a finite group, does there exist a (finite) set of sentences \( \Phi_G \) such that every model of \( \text{GT} \cup \Phi_G \) is isomorphic to \( G \)?

\textbf{Solution.} For \( \mathbb{Z}_2 \): There is (up to isomorphism) only one group with two elements. Hence it suffices to ensure that any model of \( \text{GT} \cup \Phi \) has exactly two elements. This can be done using the sentence
\[ \exists x_1, x_2 (x_1 \neq x_2 \land \forall y (y = x_1 \lor y = x_2)). \]

For \( \mathbb{Z}_4 \): There are (up to isomorphism) two groups of order four - \( \mathbb{Z}_4 \) and the Klein-Four Group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). The latter group is not cyclic. So we let, similar to the case \( \mathbb{Z}_2 \), \( \varphi_1 \) be the statement that there exist exactly 4 elements, and let \( \varphi_2 \) be the sentence
\[ \exists x \forall y (y = x \lor y = x \cdot x \lor y = x \cdot x \cdot x \lor y = x \cdot x \cdot x \cdot x). \]

Then \( \text{GT} \cup \{ \varphi_1, \varphi_2 \} \) has only \( \mathbb{Z}_4 \) as a model.

\( G \) an arbitrary finite group of order \( n \): Every finite group is completely determined (up to isomorphism) through its multiplication table. Hence we only need one sentence to define \( G \) up to isomorphism: The sentence
\[ \exists x_1, \ldots, x_n \left[ \bigwedge_{i \neq j} x_i \neq x_j \land \text{(relations of the multiplication table of the form } x_i \cdot x_j = x_k \text{)} \right]. \]

\section*{Problem 4 – The compactness theorem, again}

Fix a language \( \mathcal{L} \). Show that a set \( T \) of \( \mathcal{L} \)-sentences has a model if and only if every finite subset of \( T \) has a model.

\textbf{Solution.} Clearly, if \( T \) has a model then every finite subset of \( T \) has a model.

Now assume every finite subset of \( T \) has a model. Suppose for a contradiction \( T \) does not have a model. Then every model of \( T \) is trivially (since there is none) also a model of \( \varphi \land \neg \varphi \) for any sentence \( \varphi \), that is, \( T \models (\varphi \land \neg \varphi) \). By the completeness theorem, \( T \vdash (\varphi \land \neg \varphi) \), i.e. \( T \) is inconsistent. There must exist a finite proof of \( (\varphi \land \neg \varphi) \). This proof can use only finitely many formulas from \( T \). Collect these finitely many formulas in a finite subset \( T_0 \subseteq T \). Then \( T_0 \vdash (\varphi \land \neg \varphi) \) and hence \( T_0 \models (\varphi \land \neg \varphi) \). By assumption, \( T_0 \) has a model, say \( M \), and \( T_0 \models (\varphi \land \neg \varphi) \) implies that \( M \models (\varphi \land \neg \varphi) \), which is impossible.

\section*{Problem 5 – Non-standard models of arithmetic, part II}

Let \( \mathcal{L} = \{ S, +, \cdot, 0 \} \), and let \( \mathbb{N} \) be the standard \( \mathcal{L} \)-structure of the natural numbers.

Let \( T_{1\mathbb{N}} = \{ \varphi : \mathbb{N} \models \varphi \} \). \( T_{1\mathbb{N}} \) is called the \textit{(first-order) theory of arithmetic}. Use the compactness theorem (above, #3) to show that there exists a model of \( T_{1\mathbb{N}} \) that is not isomorphic to \( \mathbb{N} \).

\textbf{Solution.} Extend the language of arithmetic by adding a new constant symbol \( \zeta \).

For \( n \in \mathbb{N} \), let \( \varphi_n \) be the sentence \( n < \zeta \). Put \( T' = T_{1\mathbb{N}} \cup \{ \varphi_n : n \in \mathbb{N} \} \).

Every finite subset of \( T' \) has a model – the standard model \( \mathbb{N} \) (we just have to interpret \( \zeta \) large enough). By the compactness theorem in #4, we infer that \( T' \) has a model \( M \). By construction, \( M \) is also a model of \( T_{1\mathbb{N}} \). But in \( M \) it must hold that \( c \) (the interpretation of the constant symbol \( \zeta \)) is greater than every natural number.