Homework 10 for
MATH 497A, Introduction to Ramsey Theory

Due: Monday November 14

Problem 1 – Non-standard models of arithmetic, part I

Consider the language $\mathcal{L} = \{S, +, \underline{0}\}$, where S is a unary function symbol, $+$ is a binary function symbol, and $\underline{0}$ is a constant symbol.

Consider the first four Peano axioms:

(P1) $\forall x (S(x) \neq \underline{0})$

(P2) $\forall x \forall y\ (S(x) = S(y) \to x = y)$

(P3) $\forall x (x + \underline{0} = x)$

(P4) $\forall x \forall y\ x + S(y) = S(x + y)$

A structure satisfying these sentences is $\mathcal{M} = (\mathbb{N}, +1, +, 0)$, i.e. S is interpreted as adding 1, $+$ is interpreted as the usual addition of natural numbers, and $\underline{0}$ is interpreted as the number 0. Find three other (mutually non-isomorphic) structures that satisfy these sentences, but that are not isomorphic to $\mathcal{M}$.

(*Hint:* For example, you could add new elements to $\mathbb{N}$ and interpret the functions on those elements appropriately.)

Problem 2 – Models of PA

Show that $\mathbb{R}^{\geq 0} = (\mathbb{R}^{\geq 0}, +^{\mathbb{R}}, \cdot^{\mathbb{R}}, +1, 0)$ is not a model of PA.

Problem 3 – Axiomatization of groups

Let $\mathcal{L} = \{\cdot, \underline{e}\}$ be the language of groups. Find finitely many $\mathcal{L}$-sentences $\Phi = \{\varphi_1, \dots, \varphi_n\}$ such that every model of $\mathsf{GT} \cup \Phi$ is isomorphic to $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

Do the same for $\mathbb{Z}_4$.

Bonus: Is this possible for any distinct finite group? That is, if G is a finite group, does there exist a (finite) set of sentences Φ_G such that every model of $\mathsf{GT} \cup \Phi_G$ is isomorphic to G?

Problem 4 – The compactness theorem, again

Fix a language $\mathcal{L}$. Show that a set T of $\mathcal{L}$-sentences has a model if and only if every finite subset of T has a model.

Problem 5 – Non-standard models of arithmetic, part II

Let $\mathcal{L} = \{S, +, \cdot, \underline{0}\}$, and let $\mathbb{N}$ be the standard $\mathcal{L}$-structure of the natural numbers.

Let $T_{\mathbb{N}} = \{\varphi : \mathbb{N} \models \varphi\}$. $T_{\mathbb{N}}$ is called the (first-order) *theory of arithmetic*. Use the compactness theorem (above, #3) to show that there exists a model of $T_{\mathbb{N}}$ that is not isomorphic to $\mathbb{N}$.