Homework 9 for
MATH 497A, Introduction to Ramsey Theory

Due: Monday November 7

Problem 1 – Avoiding arithmetic progressions

A sequence \((a_1, a_2, \ldots, a_n)\) has a \(k\)-term monotone AP if there exists a set of indices \(\{i_1 < i_2 < \cdots < i_k\}\) such that \(a_{i_1}, a_{i_2}, \ldots, a_{i_k}\) is either an increasing or decreasing AP.

Show that for any \(n\) there exists a permutation \((\pi(1), \pi(2), \ldots, \pi(n))\) of \(\{1, \ldots, n\}\) that does not contain a monotone 3-AP.

Problem 2 – An application of the Hales-Jewett Theorem

Show that for any \(k, r \geq 1\) there exists a set \(S\) of positive integers such that \(S\) does not contain a \((k + 1)\)-AP, but for every \(r\)-coloring of \(S\) there exists a monochromatic \(k\)-AP in \(S\).

Problem 3 – Monochromatic subspaces

A subset \(S \subseteq C^n_t\) is a \(k\)-dimensional subspace if there exists a word \(w \in (A \cup \{*_1, \ldots, *_k\})^n\) in which each \(*_i\) appears at least once and so that

\[
A = \{(x_1, \ldots, x_n) \colon (x_1, \ldots, x_n) \text{ is an instantiation of } w\},
\]

where \((x_1, \ldots, x_n)\) is an instantiation of \(w\) if there exist \(y_1, \ldots, y_k\) so that if every occurrence of \(*_i\) is set to \(y_i\), we obtain \((x_1, \ldots, x_n)\).

Show that for all \(k, r, t \geq 1\) there exists a number \(N\) such that for all \(n \geq N\), if the points of \(C^n_t\) are \(r\)-colored, there exists a \(k\)-dimensional monochromatic subspace.

Problem 4 – An application of monochromatic subspaces

Show that for all \(k, r \geq 1\) there exists an \(N\) such that for any \(n \geq N\), if the power set of \(\{1, \ldots, n\}\) is \(r\)-colored, then there exist pairwise disjoint sets \(S, T_1, \ldots, T_k\) so that all sets

\[
S \cup \bigcup_{i \in I} A_i \quad I \subseteq \{1, \ldots, k\}
\]

are monochromatic.