Homework 9 for
MATH 497A, Introduction to Ramsey Theory

Due: Monday November 7

Problem 1 – Avoiding arithmetic progressions

A sequence $(a_1, a_2, \ldots, a_n)$ has a k-term monotone AP if there exists a set of indices $\{i_1 < i_2 < \cdots < i_k)$ such that $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ is either an increasing or decreasing AP.

Show that for any n there exists a permutation $(\pi(1), \pi(2), \ldots, \pi(n))$ of $\{1, \ldots, n\}$ that does not contain a monotone 3-AP.

Problem 2 – An application of the Hales-Jewett Theorem

Show that for any $k, r \geq 1$ there exists a set S of positive integers such that S does not contain a $(k+1)$-AP, but for every r-coloring of S there exists a monochromatic k-AP in S.

Problem 3 – Monochromatic subspaces

A subset $S \subseteq C_t^n$ is a *k-dimensional subspace* if there exists a word $w \in (A \cup \{*_1, \ldots, *_k\})^n$ in which each $*_i$ appears at least once and so that

$$A = \{(x_1, \ldots, x_n) \colon (x_1, \ldots, x_n) \text{ is an instantiation of } w\},$$

where $(x_1, \ldots, x_n)$ is an *instantiation* of w if there exist $y_1, \ldots, y_k$ so that if every occurrence of $*_i$ is set to y_i, we obtain $(x_1, \ldots, x_n)$.

Show that for all $k, r, t \geq 1$ there exists a number N such that for all $n \geq N$, if the points of C_t^n are r-colored, there exists a k-dimensional monochromatic subspace.

Problem 4 – An application of monochromatic subspaces

Show that for all $k, r \geq 1$ there exists an N such that for any $n \geq N$, if the power set of $\{1, \ldots, n\}$ is r-colored, then there exist pairwise disjoint sets $S, T_1, \ldots, T_k$ so that all sets

$$S \cup \bigcup_{i \in I} A_i \qquad I \subseteq \{1, \ldots, k\}$$

are monochromatic.