Homework 8 for
MATH 497A, Introduction to Ramsey Theory
Due: Monday October 31

Problem 1

Upper bounds on Van der Waerden numbers
Suppose that $k \geq 2$, $n \geq 3$. Let $S$ be the collection of set $S \subseteq [1, n]$ such that $S$ does not contain a $k$-AP. Put
$$\nu_k(n) = \max\{|S| : S \in S\}.$$ 

For fixed $k \geq 3, r \geq 2$, show that if $M_k$ is a number such that for some $m$, $\nu_k(m) \leq M_k \leq (m - 1)/r$, then
$$W(k, r) \leq r \cdot M_k + 1.$$ 

Problem 2

Lower bounds on $\nu_k(n)$
Suppose $n, k \geq 3$. Let $r(n)$ be the minimum number of colors required to color $[1, n]$ so that no monochromatic $k$-AP exists. Show that
$$\nu_k(n) \geq \left\lceil \frac{n}{r(n)} \right\rceil.$$ 

Problem 3

Lower bounds via the probabilistic method
Show, using the probabilistic method, that for large enough $k$,
$$W(k, 2) > 2^{k/2}.$$ 

Problem 4

Primitive recursive functions
Show that the following functions are primitive recursive:

(a) $f(x, y) = x \cdot y$  
(b) $g(x, y) = x^y$  
(c) $h(x) = x!$

(Give formal derivations, i.e. take the basic functions and other functions that have been shown to be primitive recursive, and use the closure properties – substitution and recursion – to obtain definitions of $f, g, h$. You can use that $r(x, y) = x + y$ is primitive recursive, as argued in class.)