Homework 8 for
MATH 497A, Introduction to Ramsey Theory

Due: Monday October 31

Problem 1

Upper bounds on Van der Waerden numbers

Suppose that $k \geq 2$, $n \geq 3$. Let $\mathcal{S}$ be the collection of set $S \subseteq [1, n]$ such that S does not contain a k-AP. Put

$$\nu_k(n) = \max\{|S| : S \in \mathcal{S}\}.$$

For fixed $k \geq 3, r \geq 2$, show that if M_k is a number such that for some m, $\nu_k(m) \leq M_k \leq (m-1)/r$, then

$$W(k, r) \leq r \cdot M_k + 1.$$

Problem 2

Lower bounds on $\nu_k(n)$

Suppose $n, k \geq 3$. Let $r(n)$ be the minimum number of colors required to color $[1, n]$ so that no monochromatic k-AP exists. Show that

$$\nu_k(n) \geq \left\lceil \frac{n}{r(n)} \right\rceil.$$

Problem 3

Lower bounds via the probabilistic method

Show, using the probabilistic method, that for large enough k,

$$W(k, 2) > 2^{k/2}.$$

Problem 4

Primitive recursive functions

Show that the following functions are primitive recursive:

$$(a) \quad f(x, y) = x \cdot y \qquad (b) \quad g(x, y) = x^y \qquad (c) \quad h(x) = x!$$

(Give formal derivations, i.e. take the basic functions and other functions that have been shown to be primitive recursive, and use the closure properties – substitution and recursion – to obtain definitions of f, g, h. You can use that $r(x, y) = x + y$ is primitive recursive, as argued in class.)