Homework 7 for
MATH 497A, Introduction to Ramsey Theory

Due: Monday October 24

Problem 1

Variant of the Compactness Principle

Let $r \geq 2$, $p \geq 1$, and let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$. Assume that for every r-coloring of $[\mathbb{N}]^p$ there exists a member $A \in \mathcal{F}$ such that $[A]^k$ is monochromatic. Show that there exists an $N > 0$ such that for every r-coloring of $[N]^p = [\{1, \dots, N\}]^p$, there exists an $A \in \mathcal{F}$ such that $A \subseteq [1, N]$ and $[A]^p$ is monochromatic.

Problem 2

Equivalent versions of Van der Waerden's Theorem

Show that the following statements are equivalent.

(i) For any $k \geq 2$, any 2-coloring of $\mathbb{N}$ admits a monochromatic AP of length k.

(ii) For any $k \geq 2$, $W(k, 2)$ exists.

(iii) For any $k, r \geq 2$, $W(k, r)$ exists.

(iv) Let $r \geq 2$. For any r-coloring of $\mathbb{N}$ and any finite subset $S = \{s_1, \dots, s_n\}$ of $\mathbb{N}$ there exist integers a, d such that $a + dS = \{a + ds_1, \dots, a + ds_n\}$ is monochromatic.

(v) For any $k, r \geq 2$, any r-coloring of $\mathbb{N}$ admits a monochromatic AP of length k.

Problem 3

Number of arithmetic progressions

Show that within the set $\{1, \dots, n\}$ there exist $\frac{n}{2(k-1)}(1 + o(1))$ arithmetic progressions of length k.

Problem 4

Lower bound

Show that $W(3, 3) > 26$.