Problem 1

**Variant of the Compactness Principle**

Let \( r \geq 2, \ p \geq 1, \) and let \( \mathcal{F} \) be a family of finite subsets of \( \mathbb{N} \). Assume that for every \( r \)-coloring of \([\mathbb{N}]^p\) there exists a member \( A \in \mathcal{F} \) such that \([A]^k\) is monochromatic. Show that there exists an \( N > 0 \) such that for every \( r \)-coloring of \([\mathbb{N}]^p = [\{1, \ldots, N\}]^p\), there exists an \( A \in \mathcal{F} \) such that \( A \subseteq [1, N] \) and \([A]^p\) is monochromatic.

Problem 2

**Equivalent versions of Van der Waerden’s Theorem**

Show that the following statements are equivalent.

(i) For any \( k \geq 2 \), any 2-coloring of \( \mathbb{N} \) admits a monochromatic AP of length \( k \).

(ii) For any \( k \geq 2 \), \( W(k, 2) \) exists.

(iii) For any \( k, r \geq 2 \), \( W(k, r) \) exists.

(iv) Let \( r \geq 2 \). For any \( r \)-coloring of \( \mathbb{N} \) and any finite subset \( S = \{s_1, \ldots, s_n\} \) of \( \mathbb{N} \) there exist integers \( a, d \) such that \( a + dS = \{a + ds_1, \ldots, a + ds_n\} \) is monochromatic.

(v) For any \( k, r \geq 2 \), any \( r \)-coloring of \( \mathbb{N} \) admits a monochromatic AP of length \( k \).

Problem 3

**Number of arithmetic progressions**

Show that within the set \( \{1, \ldots, n\} \) there exist \( \frac{n}{2(k-1)}(1 + o(1)) \) arithmetic progressions of length \( k \).

Problem 4

**Lower bound**

Show that \( W(3, 3) > 26 \).