Homework 6 for
MATH 497A, Introduction to Ramsey Theory

Due: Monday October 3

Problem 1

Failure of Ramsey’s Theorem for infinite colorings

Show that for any infinite cardinal \( \kappa \), \( 2^\kappa \not\rightarrow (3)_2^2 \).

Solution. This is in the lecture notes from 09/26.

Problem 2

Failure of Ramsey’s Theorem for power sets

Show that for any cardinal \( \kappa \), \( 2^\kappa \not\rightarrow (\kappa^+)_2^2 \).

Solution. Sketch: The strategy is the same as in the proof that \( 2^{\aleph_0} \not\rightarrow (\aleph_1)_2^2 \).

The crucial lemma is to show that the lexicographically ordered set \( \{0, 1\}^\kappa \) has no increasing or decreasing sequence of length \( \kappa^+ \).

To see this, assume for a contradiction that \( W = \{ f_\alpha : \alpha < \kappa^+ \} \subseteq \{0, 1\}^\kappa \) is a lexicographically increasing sequence of length \( \kappa^+ \) (the decreasing case is similar). Let \( \gamma \leq \kappa \) be the least \( \gamma \) such that the set \( \{ f_\alpha \upharpoonright \gamma : \alpha < \kappa^+ \} \) has size \( \kappa^+ \). Clearly \( \xi_a \) is the position where \( f_\alpha \) first differs from its successor \( f_{\alpha+1} \). To see this, assume for a contradiction that \( \gamma \leq \kappa \) is the least \( \gamma \) such that the set \( \{ f_\alpha \upharpoonright \gamma : \alpha < \kappa^+ \} \) has size \( \kappa^+ \). Hence by the infinite pigeonhole principle there exists \( \xi < \gamma \) such that \( \xi = \xi_a \) for \( \kappa^+ \) many elements \( f_\alpha \) of \( W \). However, if \( \xi = \xi_a = \xi_b \) and \( f_\alpha \upharpoonright \xi = f_b \upharpoonright \xi \), then \( f_b < f_{\alpha+1} \) and \( f_\alpha < f_{\beta+1} \), which means \( f_a = f_b \). Thus the set \( \{ f_\alpha \upharpoonright \xi : \alpha < \kappa^+ \} \) has size \( \kappa^+ \), contrary to the minimality assumption on \( \gamma \).

Problem 3

Increasing sequences of real numbers

Show that for any ordinal \( \beta < \omega_1 \), there exists an increasing sequence of reals of length \( \beta \), i.e. a sequence \( \{ a_\xi : \xi < \beta \} \) such that for any \( \xi < \beta \), \( a_\xi \triangleq a_{\xi+1} \).

Solution. We prove this by transfinite induction. For \( \beta = 0 \) there is nothing to show, so assume the assertion holds for all \( \alpha < \beta \). If \( \beta \) is a successor ordinal, \( \beta = \alpha + 1 \), let \( (a_\xi : \xi < \alpha) \) be an increasing sequence of length \( \alpha \). Using an order preserving bijection between \( \mathbb{R} \) and \( (0, 1) \), we can assume that \( (a_\xi) \) is contained in \( (0, 1) \). Then the sequence \( (b_\xi) \) defined as \( b_\xi = a_\xi \) for \( \xi < \alpha \), \( b_\alpha = 1 \) is an increasing sequence of length \( \beta \).

Now assume \( \beta \) is a countable limit ordinal. Let \( (\xi_n) \) be a sequence of countable ordinals so that \( \sup \{ \xi_n : n \in \mathbb{N} \} = \beta \).

By induction hypothesis, there exists an increasing sequence \( (a_\xi^1 : \xi < \xi_1) \) of length \( \xi_1 \). Using again an order preserving bijection between \( \mathbb{R} \) and \( (0, 1) \), we can assume that \( (a_\xi^1)_{\xi < \xi_1} \) is contained in \( (0, 1) \).
There exists a unique ordinal \( \gamma_1 \) such that \( \xi_1 + \gamma_1 = \xi_2 \). \( \gamma \) is the order type of \( \{ \eta : \xi_1 \leq \eta < \xi_2 \} \) (i.e. we “subtract” \( \xi_2 \) from \( \xi_2 \)). By induction hypothesis there exists sequence \( (b^1_\zeta : \zeta < \gamma_1) \) of length \( \gamma_1 \). Using an order preserving bijection, we can assume that \( (b^1_\zeta) \) is contained in \( (1, 2) \) and “append” it to \( (a^2_\xi) \), yielding a sequence of length \( \xi_3 \). In the limit, this yields an increasing sequence in \( (0, \infty) \) of length \( \beta \).