Homework 6 for
MATH 497A, Introduction to Ramsey Theory

Due: Monday October 3

Problem 1

Failure of Ramsey's Theorem for infinite colorings

Show that for any infinite cardinal κ, $2^\kappa \not\longrightarrow (3)^2_\kappa$.

Solution. This is in the lecture notes from 09/26. ■

Problem 2

Failure of Ramsey's Theorem for power sets

Show that for any cardinal κ, $2^\kappa \not\longrightarrow (\kappa^+)^2_2$.

Solution. *Sketch:* The strategy is the same as in the proof that $2^{\aleph_0} \not\longrightarrow (\aleph_1)^2_2$.

The crucial lemma is to show that the lexicographically ordered set $\{0,1\}^\kappa$ has no increasing or decreasing sequence of length κ^+.

To see this, assume for a contradiction that $W = \{f_\alpha \colon \alpha < \kappa^+\} \subseteq \{0,1\}^\kappa$ is a lexicographically increasing seqeunce of length κ^+ (the decreasing case is similar). Let $\gamma \le \kappa$ be the least γ such that the set $\{f_\alpha \restriction_\gamma \colon \alpha < \kappa^+\}$ has size κ^+. For each $\alpha < \kappa^+$, let ξ_α be such that $f_\alpha \restriction_{\xi_\alpha} = f_{\alpha+1} \restriction_{\xi_\alpha}$ and $f_\alpha(\xi_\alpha) = 0 \neq 1 = f_{\alpha+1}(\xi_\alpha)$, that is, ξ_α is the position where f_α first differs from its successor $f_{\alpha+1}$. Clearly $\xi_\alpha < \gamma$. There are $\gamma \le \kappa$ many choice for ξ_α. We have κ^+ many ξ_α. Hence by the infinite pigeonhole principle there exists $\xi < \gamma$ such that $\xi = \xi_\alpha$ for κ^+ many elements f_α of W. However,if $\xi = \xi_\alpha = \xi_\beta$ and $f_\alpha \restriction_\xi = f_\beta \restriction_\xi$,then $f_\beta < f_{\alpha+1}$ and $f_\alpha < f_{\beta+1}$, which means $f_\alpha = f_\beta$. Thus the set $\{f_\alpha \restriction_\xi \colon \alpha < \kappa^+\}$ has size κ^+, contrary to the minimality assumption on γ. ■

Problem 3

Increasing sequences of real numbers

Show that for any ordinal $\beta < \omega_1$, there exists an increasing sequence of reals of length β, i.e. a sequence $\{a_\xi \colon \xi < \beta\}$ such that for any $\xi < \beta$, $a_\xi <_{\mathbb{R}} a_{\xi+1}$.

Solution. We prove this by transfinite induction. For $\beta = 0$ there is nothing to show, so assume the assertion holds for all $\alpha < \beta$. If β is a successor ordinal, $\beta = \alpha + 1$, let $(a_\xi \colon \xi < \alpha)$ be an increasing sequence of length α. Using an order preserving bijection between $\mathbb{R}$ and $(0,1)$, we can assume that (a_ξ) is contained in $(0,1)$. Then the sequence (b_ξ) defined as $b_\xi = a_\xi$ for $\xi < \alpha$, $b_\alpha = 1$ is an incresing sequence of length β.

Now assume β is a countable limit ordinal. Let (ξ_n) be a sequence of countable ordinals so that $\sup\{\xi_n \colon n \in \mathbb{N}\} = \beta$.

By induction hypothesis, there exists an increasing sequence $(a^1_\zeta : \zeta < \xi_1)$ of length ξ_1. Using again an order preserving bijection between $\mathbb{R}$ and $(0,1)$, we can assume that $(a^1_\zeta)_{\zeta<\xi_1}$ is contained in $(0,1)$.

There exists a unique ordinal γ_1 such that $\xi_1 + \gamma_1 = \xi_2$. γ is the order type of $\{\eta \colon \xi_1 \leq \eta < \xi_2\}$ (i.e. we "subtract" ξ_1 from ξ_2). By induction hypothesis there exists sequence $(b^1_\zeta : \zeta < \gamma_1)$ of length γ_1. Using an order preserving bijection, we can assume that (b^1_ζ) is contained in $(1,2)$. Then the sequence $(a^2_\zeta \colon \zeta < \xi_2)$ is increasing of length ξ_2. We can now continue inductively: Let γ_2 be such that $\xi_2 + \gamma_2 = \xi_3$. Find an increasing sequence of length γ_2 in $(2,3)$ and "append" it to $(a^2\zeta)$, yielding a sequence of length ξ_3. In the limit, this yields an increasing seqeunce in $(0,\infty)$ of length β.

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Problem 4

No Banach-Tarski Paradox for $\mathbb{Z}$

An easy reformulation of the Banach-Tarski paradox says that the unit ball C in $\mathbb{R}^3$ can be *paradoxically decomposed*, i.e. there exist disjoint sets A, B such that $C = A \cup B$ and each set A, B can be further partitioned into finitely many pairwise disjoint sets $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$ such that

$$A = \bigcup_{k=1}^{m} A_k \qquad B = \bigcup_{l=1}^{n} B_l$$

and there exist rigid motions $\alpha_1, \ldots, \alpha_m$, $\beta_1, \ldots, \beta_n$ such that

$$C = \bigcup_{k=1}^{m} \alpha_k(A_k) = \bigcup_{l=1}^{n} \beta_l(B_l).$$

Show that such a decomposition is impossible for $\mathbb{Z}$, if as motions we only allow translations by integers, i.e. translations of a set A in the form $A + j = \{a + j \colon a \in A\}$ for some $j \in \mathbb{Z}$.

Solution. A proof can be found here (Proposition 3): https://terrytao.wordpress.com/2009/01/08/245b-notes-2-amenability-the-ping-pong-lemma-and-the-banach-tarski-paradox-optional/

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Problem 5

BONUS: The measure problem for non-atomic measures

Translation invariance of a measure on $\mathbb{R}$, $\mu(A + r) = \mu(A)$, implies $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$. Measures with the latter property are called *non-atomic*. Does the measure problem become solvable if we only require non-atomicity? That is, does there exist a function $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ such that

(1) $\mu([0,1]) = 1$,

(2) $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$,

(3) $\mu(\bigcup A_n) = \sum \mu(A_n)$ for any sequence of pairwise disjoint sets $A_n \subseteq \mathbb{R}$?

Solution. *Banach and Kuratowski* showed in 1929 that, under the assumption of the Continuum Hypothesis, such a measure cannot exist.

Here is a link to their original paper (in French):
http://matwbn.icm.edu.pl/ksiazki/fm/fm14/fm14110.pdf ■