Homework 5 for
MATH 497A, Introduction to Ramsey Theory

Due: Monday September 26

Problem 1 – The Axiom of Regularity

The Axiom of Regularity is formalized as

$$\text{If } S \neq \emptyset, \text{ then there exists an } x \in S, \text{ such that } x \cap S = \emptyset.$$

Show that if one assumes the Axiom of Regularity, then, for any n, there do not exist sets $x_0, x_1, \dots, x_n$ such that $x_0 \in x_1 \in \cdots \in x_n \in x_0$. Furthermore, there does not exist a set X with an infinite descending $\in$-sequence, i.e. a sequence $(x_i)_{i \in \mathbb{N}}$ such that

$$X \ni x_0 \ni x_1 \ni x_2 \ni \dots.$$

Solution. We first show that it is impossible that $x \in x$. Otherwise, the set $S = x \cap \{x\}$ is not empty, but for every $y \in S$, $y \cap S = x$ and x is non-empty, contradicting Regularity.

To see that $x_0 \in x_1 \in x_0$ is impossible, apply the above argumentation to $S = \{x_1, x_0\}$, etc.

Finally, assume there exists $X \ni x_0 \ni x_1 \ni x_2 \ni \dots$. Consider the sets $X' = \{x_n : n \geq 0\}$. By regularity, there must exists $y \in X'$ such that $y \cap X' = \emptyset$. But $y \in X'$ means that $y = x_n$ for some n, and hence $x_{n+1} \in x_n \cap X'$, contradiction. ∎

Problem 2 – Properties of Ordinals

Show that if α is an ordinal, then $\alpha \cup \{\alpha\}$ is an ordinal. Show further that if X is a non-empty set of ordinals, then $\bigcup X = \{\beta : \exists \alpha \in X \text{ with } \beta \in \alpha\}$ is an ordinal, and $\bigcup X = \sup X$.

Solution. Suppose α is an ordinal. To see that $\alpha + 1 = \alpha \cup \{\alpha\}$ is transitive, suppose $x \in \alpha + 1$. Then either $x \in \alpha$, in which case it follows from the transitivity of α that $x \subseteq \alpha$, or $x = \alpha$, in which case we have $x = \alpha \subseteq \alpha \cup \{\alpha\}$. Furthermore, by Regularity it suffices to show that $\alpha + 1$ is linearly ordered by $\in$. By assumption, α is linearly ordered by $\in$. Suppose $x, y \in \alpha + 1$, $x \neq y$. If $x, y \in \alpha$ then we know that either $x \in y$ or $y \in x$. If one of them, say y, is not in α, then $y = \alpha$, and hence $x \in y$ holds. Therefore, any two elements are comparable.

Now suppose X is a non-empty set of ordinals. We first argue that $\bigcup X$ is transitive: Let $x \in \bigcup X$. Then there exists an ordinal $\alpha \in X$ such that $x \in \alpha$. Since α is transitive, $x \subseteq \alpha$. Hence, by definition of $\bigcup X$, $x \in \bigcup X$. To see that $\bigcup X$ is linearly ordered by $\in$, let $x, y \in \bigcup X$, $x \neq y$. Let $\alpha, \beta \in X$ such that $x \in \alpha$, $y \in \beta$. Since α, β are ordinals, we either have $\alpha \in \beta$, $\alpha = \beta$, or $\beta \in \alpha$. Say $\alpha \in \beta$. (The other two cases are similar.) Since β is transitive, we have $x \in \beta$. Since β is linearly ordered by $\in$, we have $x \in y$ or $x = y$ or $y \in x$, as required.

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Problem 3 – The ordinal ε_0

Let $\varepsilon_0 = \lim_{n\to\omega} \alpha_n = \bigcup\{\alpha_n : n \in \omega\}$, where $\alpha_0 = \omega$ and $\alpha_{n+1} = \omega^{\alpha_n}$.

Show that ε_0 is the least ordinal ε so that $\omega^\varepsilon = \varepsilon$.

Solution. We have

$$\omega^{\varepsilon_0} = \sup\{\omega^\beta : \beta < \varepsilon_0\} = \sup\{\omega^{\alpha_n} : n \in \mathbb{N}\} = \sup\{\alpha_n : n \in \mathbb{N}\} = \varepsilon.$$

Hence ε has the fixed-point property. Suppose now that there exists an ordinal $\gamma < \varepsilon$ with $\omega^\gamma = \gamma$. Let n be so that $\alpha_n \gamma < \alpha_{n+1} < \varepsilon$. Then

$$\alpha_{n+1} = \omega^{\alpha_n} \leq \omega^\gamma = \gamma < \alpha_{n+1},$$

contradiction. ■

Problem 4 – Zorn's Lemma and the Axiom of Choice

Show that Zorn's Lemma (ZL) implies the Axiom of Choice (AC). We will see in class that AC implies ZL. Hence the two principles are equivalent.

Solution. Let S be a family of nonempty sets. To find a choice function on S, let $P = \{f : f$ is a choice function on some $Z \subset S\}$. We order P by letting $f < g$ if $\mathrm{dom}(f) \subset \mathrm{dom}(g)$ and $g|_{\mathrm{dom}(f)} = f$. Now every chain C in P has an upper bound: Simply consider the function F with domain $\bigcup_C \mathrm{dom}(f)$ and $F|_{\mathrm{dom}(f)} = f$ for any $f \in C$. (The chain condition ensures that F is well-defined.)

By Zorn's Lemma, P has a maximal element G. Then G must be a choice function on all of S, for otherwise we could extend the domain of G to make it a larger choice function, contradicting maximality. ■

Problem 5 – Partial orders and linear orders

Show that every finite partial order $(P, <)$ can be extended to a linear order $(P, <')$, i.e. there exists a linear order $<'$ on P such that for all $x, y \in P$, $x < y$ implies $x <' y$.

Solution. Proceed by induction on the cardinality of S. For $|S| = 0$ there is nothing to prove. Assume the assertion holds for all set of cardinality $\leq n$, and let S be a partially ordered set of cardinality $n+1$. Since a partial order cannot contain cycles, there must exist an element s_0 such that

$$(\forall s \in S)[s \leq s_0 \;\rightarrow\; s = s_0].$$

In other words, no element of S is below s_0.

Let $S' = S \setminus \{s_0\}$. By induction hypothesis, we can extend $\leq$ to a total (linear) order on S'. Now setting $s_0 \leq s$ for all $s \in S$ completes this into a total order on S. ■