Homework 5 for
MATH 497A, Introduction to Ramsey Theory

Due: Monday September 26

Problem 1 – The Axiom of Regularity

The Axiom of Regularity is formalized as

$$\text{If } S \neq \emptyset\text{, then there exists an } x \in S\text{, such that } x \cap S = \emptyset.$$

Show that if one assumes the Axiom of Regularity, then, for any n, there do not exist sets $x_0, x_1, \dots, x_n$ such that $x_0 \in x_1 \in \cdots \in x_n \in x_0$. Furthermore, there does not exist a set X with an infinite descending $\in$-sequence, i.e. a sequence $(x_i)_{i\in\mathbb{N}}$ such that

$$X \ni x_0 \ni x_1 \ni x_2 \ni \dots.$$

Problem 2 – Properties of Ordinals

Show that if α is an ordinal, then $\alpha \cup \{\alpha\}$ is an ordinal. Show further that if X is a non-empty set of ordinals, then $\bigcup X = \{\beta : \exists \alpha \in X \text{ with } \beta \in \alpha\}$ is an ordinal, and $\bigcup X = \sup X$.

Problem 3 – The ordinal ε_0

Let $\varepsilon_0 = \lim_{n\to\omega} \alpha_n = \bigcup\{\alpha_n : n \in \omega\}$, where $\alpha_0 = \omega$ and $\alpha_{n+1} = \omega^{\alpha_n}$.

Show that ε is the least ordinal ε so that $\omega^{\varepsilon} = \varepsilon$.

Problem 4 – Zorn's Lemma and the Axiom of Choice

Show that Zorn's Lemma (ZL) implies the Axiom of Choice (AC). We will see in class that AC implies ZL. Hence the two principles are equivalent.

Problem 5 – Partial orders and linear orders

Show that every finite partial order $(P, <)$ can be extended to a linear order $(P, <')$, i.e. there exists a linear order $<'$ on P such that for all $x, y \in P$, $x < y$ implies $x <' y$.