Homework 4 for **MATH 497A**, Introduction to Ramsey Theory

Due: Monday September 19

Problem 1

Upper Bounds for Ramsey's Theorem

From the various proofs of Ramsey's Theorem, try to extract an upper bound (as sharp as you can) on $R(p;k;r)$. Recall that $R(p;k;r)$ is the least N such that

$$N \rightarrow (k)^p_r.$$

Problem 2

Failure of Ramsey's Theorem for ω-subsets

If X is an infinite set, let $[X]^\omega$ be the set of denumerable subsets of X, i.e. $[X]^\omega = \{A \subseteq X : A \text{ is countably infinite}\}$. Show that for any infinite set X there exists a 2-coloring c of $[X]^\omega$ with no infinite homogenous set.

Solution. To make the problem a little easier, we restrict ourselves to the case $X = \mathbb{N}$. The general theorem follows by using the well-ordering principle and working with cardinals.

An element of $[\mathbb{N}]^\omega$ is of the form $x = \{x_1 < x_2 < x_3, \dots\}$ with each x_i in $\mathbb{N}$. For each $x, y \in [\mathbb{N}]^\omega$, define $x \sim y$ iff $x_i = y_i$ for all but finitely many i. This defines an equivalence relation on $[\mathbb{N}]^\omega$. Pick an element from each equivalence class. Define a 2-coloring of $[\mathbb{N}]^\omega$ by letting $c(x) = 0$ if x differs from the representative of its class on an even number of positions, and $c(x) = 1$ if x differs from the representative of its class on an odd number of positions.

Let $H = \{h_1 < h_2 < h_3 < \dots\} \subseteq \mathbb{N}$ be infinite. We show that H cannot be homogenous for c. Consider the set $H_{1/2} = \{h_2 < h_4 < h_6 < \dots\}$. Let $K \in [\mathbb{N}]^\omega$ be the representative of the equivalence class of $H_{1/2}$. Let m be the least natural number from which on $H_{1/2}$ and K agree, i.e. for all $n \geq m$, $h_{2n} = k_n$. Then $H' = \{h_2 < h_4 < \dots < h_{2(n-1)} < h_{2n+1} < h_{2(n+1)} < h_{2(n+2)} < \dots\}$ is equivalent to K, but differs from it on one more position than $H_{1/2}$

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Problem 3

Cardinalities

Show that $|\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$. Show further, without using the Cantor-Schröder-Bernstein Theorem, that $|(0,1)| = |[0,1]|$.

Solution. To see $|\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$, identify a subset A of $\mathbb{N}$ with its characteristic sequence $c_A \in \{0,1\}^\mathbb{N}$ and define $f : \{0,1\}^\mathbb{N} \to \mathbb{R}$ by

$$f(x) = \sum_{i=1}^{\infty} \frac{2x_i}{3^i}.$$

This defines an injection from $\{0,1\}^\mathbb{N}$ into $[0,1]$. (The image of f is known as the *Middle-Third Cantor Set*.) To see this, assume $x, y \in \{0,1\}^\mathbb{N}$, $x \neq y$. Let n be minimal such that $x_n \neq y_n$. Wlog $x_n = 0$, $y_n = 1$. Then

$$f(y) - f(x) = \frac{2}{3^n} - \sum i = n+1^\infty \frac{2(y_i - x_i)}{3^i} \geq \frac{2}{3^n} - \sum i = n+1^\infty \frac{1}{3^i} = \frac{2}{3^n} - \frac{1}{2 \cdot 3^n} > 0.$$

We define two auxiliary mappings $f_l(0) = 1/3$ and $f_l(1/n) = 1/(n+1)$ for $n \geq 3$. Similarly, $f_r(1) = 2/3$ and $f_r(n - 1/n) = n/(n+1)$ for $n/geq3$. Then the function

$$f(x) = \begin{cases} f_l(x) & x = 0 \text{ or } x = 1/n \text{ for } n \geq 3, \\ f_r(x) & x = 1 \text{ or } x = n - 1/n \text{ for } n \geq 3, \\ x & \text{otherwise,} \end{cases}$$

is a bijection between $[0,1]$ and $(0,1)$. ■

Problem 4

Uncountabiliy of the Real Numbers

Use the completeness of $\mathbb{R}$ to give a different proof of its uncountability: For every sequence (a_n) of real numbers and for any non-empty interval I, there exists a point $p \in I$ such that $p \neq a_n$ for all n. Use completeness in the following form:

> For any nested sequence $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ of closed, non-empty intervals in $\mathbb{R}$, the intersection $\bigcap_n I_n$ is not empty.

(Do you need the Axiom of Choice here?)

Solution. The following proof works without using the Axiom of Choice.

Split I into three equal closed subintervals (overlapping at the endpoints only). Let I_1 be the leftmost part which does not contain a_1. Now assume we have constructed $I_1 \supset I_2 \supset \cdots \supset I_n$ such that $a_i \notin I_i$, which then implies $\{a_1, \dots a_n\} \cap I_n = \emptyset$. Split I_n into three equal parts and let I_{n+1} be the leftmost part that does not contain a_{n+1}.

By induction we obtain a nested sequence of closed intervals $I_1 \supset I_2 \supset I_3 \supset \dots$ so that for each n, $a_n \notin I_n$. By completeness, the intersection $\bigcap I_n$ is not empty and, by construction, cannot contain any point a_n. ■

Problem 5

Isomorphism of Linear Orders

A linear order $(P, <)$ is *dense* if for any $x, y \in P$ with $x < y$ there exists $z \in P$ such that $x < z < y$. Moreover, $(P, <)$ has *no endpoints* if for any $x \in P$ there exists a $y, z \in P$ such that $y < x < z$.

Show that any infinite countable, dense linear order with no endpoints is isomorphic to $\mathbb{Q}$ (with the standard ordering).

Solution. The standard proof of this is known as the *back and forth method*.

See the [Wikipedia entry with proof]

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