# Homework 4 for
# **MATH 497A**, Introduction to Ramsey Theory

Due: Monday September 19

**Problem 1**

**Upper Bounds for Ramsey's Theorem**

From the various proofs of Ramsey's Theorem, try to extract an upper bound (as sharp as you can) on $R(p;k;r)$. Recall that $R(p;k;r)$ is the least $N$ such that

$$N \rightarrow (k)^p_r.$$

**Problem 2**

**Failure of Ramsey's Theorem for $\omega$-subsets**

If $X$ is an infinite set, let $[X]^\omega$ be the set of denumerable subsets of $X$, i.e. $[X]^\omega = \{A \subseteq X : A \text{ is countably infinite}\}$. Show that for any infinite set $X$ there exists a 2-coloring $c$ of $[X]^\omega$ with no infinite homogenous set.

**Problem 3**

**Cardinalities**

Show that $|\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$. Show further, without using the Cantor-Schröder-Bernstein Theorem, that $|(0,1)| = |[0,1]|$.

**Problem 4**

**Uncountabiliy of the Real Numbers**

Use the completeness of $\mathbb{R}$ to give a different proof of its uncountability: For every sequence $(a_n)$ of natural numbers and for any non-empty interval $I$, there exists a point $p \in I$ such that $p \neq a_n$ for all $n$. Use completeness in the following form:

> For any nested sequence $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ of closed, non-empty intervals in $\mathbb{R}$, the intersection $\bigcap_n I_n$ is not empty.

(Do you need the Axiom of Choice here?)

**Problem 5**

**Isomorphism of Linear Orders**

A linear order $(P,<)$ is *dense* if for any $x, y \in P$ with $x < y$ there exists $z \in P$ such that $x < z < y$. Moreover, $(P,<)$ has *no endpoints* if for any $x \in P$ there exists a $y, z \in P$ such that $y < x < z$.

Show that any infinite countable, dense linear order with no endpoints is isomorphic to $\mathbb{Q}$ (with the standard ordering).