Homework 3 for
MATH 497A, Introduction to Ramsey Theory

Due: Monday September 12

Problem 1

A geometric application of Turán's Theorem.

Let $S \subseteq \mathbb{R}^2$ with d the usual Euclidean distance. The *diameter* of S is given by

$$d(S) = \sup\{d(x, y) \colon x, y \in S\}.$$

Assume now $S = \{x_1, x_2, \dots, x_n\}$ and $d(S) \le 1$. Show that the maximum number of pairs of points x, y in S with $d(x, y) > 1/\sqrt{2}$ is $\lfloor n^2/3 \rfloor$.

Show further that this bound is sharp by exhibiting, for each n, a set of diameter 1 with exactly $\lfloor n^2/3 \rfloor$ pairs of points at distance $> 1/\sqrt{2}$.

Solution. Define a graph on $\{x_1, \dots, x_n\}$ by putting

$$\{x_i, x_j\} \in E \quad \Leftrightarrow \quad d(x_i, x_j) > 1/\sqrt{2}.$$

We show that this graph does not contain a 4-clique, which implies by Turán's Theorem that $|E| \le n^2/3$, and hence that at most $\lfloor n^2/3 \rfloor$ pairs of points have distance $> 1/\sqrt{2}$.

Assume for a contradiction $x_i, x_j, x_k, x_l \in S$ form a 4-clique. It is not hard to see that three of the points, say x_i, x_j, x_k must form an angle of at least 90°. This implies

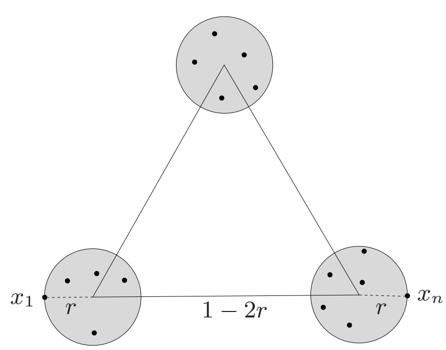
$$d(x_i, x_k) \ge \sqrt{d(x_i, x_j)^2 + d(x_j, x_k)^2} > \sqrt{(1/\sqrt{2})^2 + (1/\sqrt{2})^2} = 1,$$

which contradicts the assumption $d(S) \le 1$.

From Bonday and Murty, *Graph Theory*, 2008:

One can construct a set $\{x_1, x_2, \dots, x_n\}$ of diameter 1 in which exactly $\lfloor n^2/3 \rfloor$ pairs of points at distance $> 1/\sqrt{2}$ as follows:

Choose r such that $0 < r < (1 - 1/\sqrt{2})/4$ and draw three circles of radius r whose centres are at distance $1 - 2r$ from another. Set $p = \lfloor n/3 \rfloor$. Place points $x_1, \dots, x_p$ in one circle, points $x_{p+1}, \dots, x_{2p}$ in another, and $x_{2p+1}, \dots, x_n$ in the third.



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Problem 2

An Anti-Ramsey Theorem.
The infinite Ramsey Theorem says that, for any $p, r \geq 1$, if we color the set $[\mathbb{N}]^p$ with r colors, then there exists an infinite $H \subseteq \mathbb{N}$ so that the coloring is monochromatic on $[H]^p$.

Perhaps a bit ironically, one can use Ramsey's Theorem to prove the following "Anti"-Ramsey-Theorem:
Let $p \geq 1$, $f : [\mathbb{N}]^p \to \mathbb{N}$. Further assume there is a number $M \in \mathbb{N}$ so that for each $i \in \mathbb{N}$, $|\{x \in [\mathbb{N}]^p : f(x) = i\}| \leq M$. Show that there exists an infinite $H \subseteq \mathbb{N}$ such that f is one-one on $[H]^p$.

(*Hint:* Enumerate all elements of $[\mathbb{N}]^p$. (This is a countable set!) Define a coloring on $[\mathbb{N}]^p$ that measures how many predecessors of $\{x_1, \dots, x_p\} \in [\mathbb{N}]^p$ have the same color as $\{x_1, \dots, x_p\}$. Use Ramsey's Theorem for this coloring.)

Solution. Let $z_1, z_2, z_3, \dots$ be an enumeration of $[\mathbb{N}]^p$. Define a coloring on $[\mathbb{N}]^p$ by

$$c(z_i) = |\{j < i : f(z_j) = f(z_i)\}|.$$

By the assumption on f, this is an M-coloring of $[\mathbb{N}]^p$. The infinite Ramsey Theorem applied to c gives us an infinite homogeneous subset H of $\mathbb{N}$. For this homogeneuos set H, no two distinct elements $z_i, z_j \in [H]^p$ can have the same f-value: either $j < i$, in which case $c(z_j) < c(z_i)$, or $i < j$, in which case $c(z_i) < c(z_j)$. ■