Homework 3 for
MATH 497A, Introduction to Ramsey Theory

Due: Monday September 12

Problem 1

A geometric application of Turán's Theorem.

Let $S \subseteq \mathbb{R}^2$ with d the usual Euclidean distance. The *diameter* of S is given by

$$d(S) = \sup\{d(x,y) \colon x, y \in S\}.$$

Suppose now $S = \{x_1, x_2, \ldots, x_n\}$ and $d(S) \leq 1$. Show that the maximum number of pairs of points x, y in S with $d(x,y) > 1/\sqrt{2}$ is $\lfloor n^2/3 \rfloor$.

Show further that this bound is sharp by exhibiting, for each n, a set of diameter 1 with exactly $\lfloor n^2/3 \rfloor$ pairs of points at distance $> 1/\sqrt{2}$.

Problem 2

An Anti-Ramsey Theorem.

The infinite Ramsey Theorem says that, for any $p, r \geq 1$, if we color the set $[\mathbb{N}]^p$ with r colors, then there exists an infinite $H \subseteq \mathbb{N}$ so that the coloring is monochromatic on $[H]^p$.

Perhaps a bit ironically, one can use Ramsey's Theorem to prove the following "Anti"-Ramsey-Theorem:

Let $p \geq 1$, $f : [\mathbb{N}]^p \to \mathbb{N}$. Further assume there is a number $M \in \mathbb{N}$ so that for each $i \in \mathbb{N}$, $|\{x \in [\mathbb{N}]^p \colon f(x) = i\}| \leq M$. Show that there exists an infinite $H \subseteq \mathbb{N}$ such that f is one-one on $[H]^p$.

(*Hint:* Enumerate all elements of $[\mathbb{N}]^p$. (This is a countable set!) Define a coloring on $[\mathbb{N}]^p$ that measures how many predecessors of $\{x_1, \ldots, x_p\} \in [\mathbb{N}]^p$ have the same color as $\{x_1, \ldots, x_p\}$. Use Ramsey's Theorem for this coloring.)