Homework 2 for
MATH 497A, Introduction to Ramsey Theory

Due: **Wednesday** September 7

Problem 1

Show that the Ramsey numbers $R(m,n)$ (really $R(m,n;2)$ in light of Problem 2) satisfy the bound

$$R(m,n) \le \binom{m+n-2}{m-1}$$

for all $m,n \ge 1$. (*Hint:* Exploit the familiar recursions for the binomial coefficients)

Solution. We prove the assertion by simultaneous induction on m,n. To ground the induction note that

$$R(m,2) = m = \binom{m}{m-1} = \binom{m+2-2}{m-1} \text{ and } R(2,n) = n = \binom{n}{1} = \binom{2+n-2}{2-1}.$$

Now assume the assertion has been proved for all (k,n), $k < m$, and (m,l), $l < n$. Then

$$R(m,n) \le R(m-1,n) + R(m,n-1) \le \binom{m+n-3}{m-2} + \binom{m+n-3}{m-1} = \binom{m+n-2}{m-1}.$$

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Show further that $R(k)(= R(k,k) = R(k,k;2)) \le 2^{2k-3}$. We have

$$R(k) \le \binom{2k-2}{k-1} \le \binom{2k-3}{k-1} + \binom{2k-3}{k-2} \le 2^{2k-3},$$

since $\sum \binom{n}{k} = 2^k$.

Problem 2

Prove Ramsey's Theorem for r colors. That is, show that for any $k \ge 1$ and any $r \ge 1$ there exists a number $R(k;r) = R(k,k;r)$ such that whenever $G = (V,E)$ is a graph on $\ge R(k,k;r)$ vertices, and $c : E \to \{1,\ldots,r\}$ is an r-coloring of the edges of G, then there exists j, $1 \le j \le r$ and $W \subseteq V$ such that $c(e) = c_j$ for all edges connecting two vertices in W.

Problem 3

Show that if the integer plane $\mathbb{Z}^2 = \{(x,y) : x,y \in \mathbb{Z}\}$ is 2-colored, there exists a monochromatic rectangle. i.e. a rectangle with all four corners the same color. Can you generalize this result to r colors?

Solution. We prove the general result for r colors. Consider the grid given by $1 \le x \le r+1$ and $1 \le y \le r^{r+1}+1$. Each row corresponds to an r-coloring of the set $\{1,\ldots,r+1\}$. There are r^{r+1} different colorings, so within the grid one row must have the same coloring. Since a row in the grid contains $r+1$ elements, one color must appear at least twice in both rows (at the same position, respectively). This gives rise to a monochromatic rectangle. ■

Nota Bene: If you like this problem, you may find this challenge interesting –
`http://blog.computationalcomplexity.org/2009/11/17x17-challenge-worth-28900-this-is-not.html`

Problem 4

Complete the following, alternative proof of Turán's Theorem:

Proceed by induction on $N = |V|$. Assume the assertion is proven for $N-1$. Suppose $G = (V,E)$ is a graph on N vertices without a k-clique with a maximal number of edges (i.e. if we add one more edge, we have get a k-clique). Argue first that G contains a $(k-1)$-clique. Let $A \subseteq V$ be such a clique, and let $B = V \setminus A$. Now obtain upper bounds on (1) the number of edges between vertices in A, (2) the number of edges connecting A and B, (3) the number of edges between vertices in B. Add up the three upper bounds to obtain the desired upper bound on $|E|$.

Solution. If G did not contain a $(k-1)$-clique, there would be two vertices of degree $k-2$, and hence we could add an edge without creating a k-clique.

Now we have: (1) the number of edges e_A in A is $\binom{k-1}{2}$. (2) No vertex in B can be adjacent to more than $k-2$ vertices in A, since other wise this vertex and A would form a k-clique. Hence $e_{AB} \leq (k-2)(N-k+1)$. (3) The number of vertices in B is less than N, and so we can use the induction hypothesis and conclude $e_B \leq (1-\frac{1}{k-1})\frac{(N-k+1)^2}{2}$.

Putting the three bounds together we obtain

$$\begin{aligned}
|E| &\leq \binom{k-1}{2} + (k-2)(N-k+1) + \left(1-\frac{1}{k-1}\right)\frac{(N-k+1)^2}{2} \\
&= \frac{(k-1)(k-2)}{2} + \frac{k-2}{k-1}(N-k+1)(k-1) + \left(1-\frac{1}{k-1}\right)\frac{(N-k+1)^2}{2} \\
&= \frac{(k-2)}{k-1}\frac{(k-1)^2}{2} + \frac{k-2}{k-1}(N-k+1)(k-1) + \left(1-\frac{1}{k-1}\right)\frac{(N-k+1)^2}{2} \\
&= \left(1-\frac{1}{k-1}\right)\frac{((k-1)+(N-k+1))^2}{2} \\
&= \left(1-\frac{1}{k-1}\right)\frac{N^2}{2}.
\end{aligned}$$

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