Homework 2 for MATH 497A, Introduction to Ramsey Theory

Due: Wednesday September 7

Problem 1

Show that the Ramsey numbers $R(m, n)$ (really $R(m, n; 2)$ in light of Problem 2) satisfy the bound

$$R(m, n) \leq \binom{m + n - 2}{m - 1}$$

for all $m, n \geq 1$. (Hint: Exploit the familiar recursions for the binomial coefficients)

Solution. We prove the assertion by simultaneous induction on $m, n$. To ground the induction note that

$$R(m, 2) = m = \binom{m}{m - 1} = \binom{m + 2 - 2}{m - 1}$$

and

$$R(2, n) = n = \binom{n}{1} = \binom{2 + n - 2}{1 + 1}.$$

Now assume the assertion has been proved for all $(k, n), k < m$, and $(m, l), l < n$. Then

$$R(m, n) \leq R(m - 1, n) + R(m, n - 1) \leq \binom{m + n - 3}{m - 2} + \binom{m + n - 3}{m - 1} = \binom{m + n - 2}{m - 1}.$$

Show further that $R(k) (= R(k, k) = R(k, k; 2)) \leq 2^{2k - 3}$. We have

$$R(k) \leq \binom{2k - 2}{k - 1} \leq \binom{2k - 3}{k - 1} + \binom{2k - 3}{k - 2} \leq 2^{2k - 3},$$

since $\sum \binom{n}{k} = 2^k$.

Problem 2

Prove Ramsey’s Theorem for $r$ colors. That is, show that for any $k \geq 1$ and any $r \geq 1$ there exists a number $R(k; r) = R(k, k; r)$ such that whenever $G = (V, E)$ is a graph on $\geq R(k, k; r)$ vertices, and $c : E \to \{1, \ldots, r\}$ is an $r$-coloring of the edges of $G$, then there exists $j, 1 \leq j \leq r$ and $W \subseteq V$ such that $c(e) = c_j$ for all edges connecting two vertices in $W$.

Problem 3

Show that if the integer plane $\mathbb{Z}^2 = \{(x, y) : x, y \in \mathbb{Z}\}$ is 2-colored, there exists a monochromatic rectangle. i.e. a rectangle with all four corners the same color. Can you generalize this result to $r$ colors?

Solution. We prove the general result for $r$ colors. Consider the grid given by $1 \leq x \leq r + 1$ and $1 \leq y \leq r^{r+1} + 1$. Each row corresponds to an $r$-coloring of the set $\{1, \ldots, r + 1\}$. There are $r^{r+1}$ different colorings, so within the grid one row must have the same coloring. Since a row in the grid contains $r + 1$ elements, one color must appear at least twice in both rows (at the same position, respectively). This gives rise to a monochromatic rectangle.

Nota Bene: If you like this problem, you may find this challenge interesting –

Problem 4

Complete the following, alternative proof of Turán’s Theorem:

Proceed by induction on \( N = |V| \). Assume the assertion is proven for \( N - 1 \). Suppose \( G = (V,E) \) is a graph on \( N \) vertices without a \( k \)-clique with a maximal number of edges (i.e. if we add one more edge, we have get a \( k \)-clique). Argue first that \( G \) contains a \((k-1)\)-clique. Let \( A \subseteq V \) be such a clique, and let \( B = V \setminus A \). Now obtain upper bounds on (1) the number of edges between vertices in \( A \), (2) the number of edges connecting \( A \) and \( B \), (3) the number of edges between vertices in \( B \). Add up the three upper bounds to obtain the desired upper bound on \(|E|\).

Solution. If \( G \) did not contain a \((k-1)\)-clique, there would be two vertices of degree \( k - 2 \), and hence we could add an edge without creating a \( k \)-clique.

Now we have: (1) the number of edges \( e_A \) in \( A \) is \((k-1)\). (2) No vertex in \( B \) can be adjacent to more than \( k - 2 \) vertices in \( A \), since otherwise this vertex and \( A \) would form a \( k \)-clique. Hence \( e_{AB} \leq (k-2)(N-k+1) \).

(3) The number of vertices in \( B \) is less than \( N \), and so we can use the induction hypothesis and conclude \( e_B \leq (1 - \frac{1}{k-1}) \frac{(N-k+1)^2}{2} \).

Putting the three bounds together we obtain

\[
|E| \leq \left( \frac{k-1}{2} \right) + (k-2)(N-k+1) + \left( 1 - \frac{1}{k-1} \right) \frac{(N-k+1)^2}{2} = \frac{(k-1)(k-2)}{2} + \frac{k-2}{k-1}(N-k+1)(k-1) + \left( 1 - \frac{1}{k-1} \right) \frac{(N-k+1)^2}{2} = \left( 1 - \frac{1}{k-1} \right) \frac{(k-1) + (N-k+1))^2}{2} = \left( 1 - \frac{1}{k-1} \right) \frac{N^2}{2}.
\]

■