Homework 2 for
MATH 497A, Introduction to Ramsey Theory

Due: **Wednesday** September 7

Problem 1

Show that the Ramsey numbers $R(m,n)$ (really $R(m,n;2)$ in light of Problem 2) satisfy the bound
$$R(m,n) \leq \binom{m+n-2}{m-1}$$
for all $m,n \geq 1$. (*Hint:* Exploit the familiar recursions for the binomial coefficients)

Show further that $R(k)(= R(k,k) = R(k,k;2)) \leq 2^{2k-3}$.

Problem 2

Prove Ramsey's Theorem for r colors. That is, show that for any $k \geq 1$ and any $r \geq 1$ there exists a number $R(k;r) = R(k,k;r)$ such that whenever $G = (V,E)$ is a graph on $\geq R(k,k;r)$ vertices, and $c : E \to \{1,\ldots,r\}$ is an r-coloring of the edges of G, then there exists j, $1 \leq j \leq r$ and $W \subseteq V$ such that $c(e) = c_j$ for all edges connecting two vertices in W.

Problem 3

Show that if the integer plane $\mathbb{Z}^2 = \{(x,y)\colon x,y \in \mathbb{Z}\}$ is 2-colored, there exists a monochromatic rectangle. i.e. a rectangle with all four corners the same color. Can you generalize this result to r colors?

Nota Bene: If you like this problem, you may find this challenge interesting –
`http://blog.computationalcomplexity.org/2009/11/17x17-challenge-worth-28900-this-is-not.html`

Problem 4

Complete the following, alternative proof of Turán's Theorem:

Proceed by induction on $N = |V|$. Assume the assertion is proven for $N-1$. Suppose $G = (V,E)$ is a graph on N vertices without a k-clique with a maximal number of edges (i.e. if we add one more edge, we have get a k-clique). Argue first that G contains a $(k-1)$-clique. Let $A \subseteq V$ be such a clique, and let $B = V \setminus A$. Now obtain upper bounds on (1) the number of edges between vertices in A, (2) the number of edges connecting A and B, (3) the number of edges between vertices in B. Add up the three upper bounds to obtain the desired upper bound on $|E|$.