Homework 1 for
MATH 497A, Introduction to Ramsey Theory

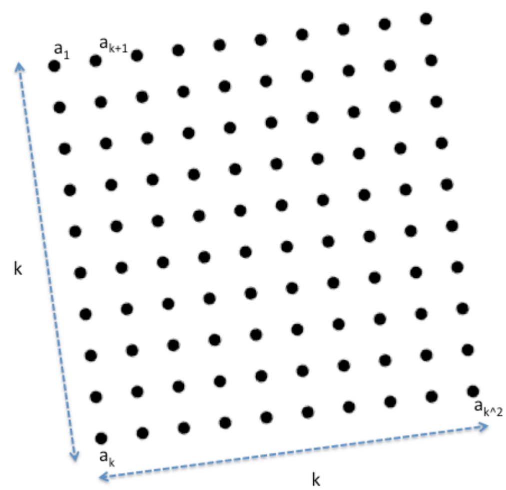
Solutions

Problem 1

Use Ramsey's theorem for graphs to show that for every positive integer k there exists a number $N(k)$ such that if $a_1, a_2, \ldots, a_{N(k)}$ is a sequence of $N(k)$ integers, it has a non-increasing subsequence of length k or a non-decreasing subsequence of length k. Show that $N(k+1) > k^2$.

Solution. Consider the graph $K^{R(k)}$, where $R(k)$ is the k-th Ramsey number. Color the edges of $K^{R(k)}$ as follows: If $i < j$ and $a_i \leq a_j$, let the edge be red, otherwise, let it be blue. By Ramsey's theorem, $K^{R(k)}$ has a complete monochromatic subgraph on k vertices $\{v_1, \ldots, v_k\} \subseteq \{1, \ldots, R(k)\}$, where $i < j$ implies $v_i < v_j$. If this subgraph is red, it follows from the definition of the coloring that $a_{v_1}, \ldots, a_{v_k}$ is a non-decreasing sequence. If the subgraph is blue, it is a non-increasing one.

To see that $N(k+1) > k^2$, consider a square grid of k^2 points, e.g. the set $\{(a,b) \in \mathbb{Z}^2 : 0 \leq a, b \leq k-1\}$. Tilt the grid lightly counterclockwise and form a sequence by enumerating the second coordinates of the points according to their first coordinate, i.e. from left to right.



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Problem 2

Use Ramsey's theorem for graphs to show that for every positive integer k there exists a number $M(k)$ such that if the set $\{1, 2, \ldots, M(k)\}$ is partitioned into two subsets, at least one of them contains a set of the form $\{x_1, x_2, \ldots, x_k, x_1 + \cdots + x_k\}$.

(*Hint:* Consider a complete graph on vertices $\{0, 1, 2, \ldots, M\}$, where M is an integer. Devise a 2-coloring of the graph so that a complete monochromatic subgraph on $k+1$ vertices yields the desired set.)

Solution. Let $M = M(k) = R(k+1)$. Suppose $c : \{1, \ldots, M\} \to \{\text{blue,red}\}$ is a 2-coloring. Consider the complete graph K^{M+1} on $M+1$ vertices. Define 2-coloring c' of the edges of K^{M+1} by putting $c'(\{i,j\}) = c(|i-j|)$. Since $M = R(k+1)$, K^{M+1} must have a complete monochromatic subgraph on $k+1$ edges. Suppose $\{v_1 < v_2 < \cdots < v_{k+1}\}$ are the vertices of such a subgraph, listed in increasing order. For $i \leq k$, put $x_i = v_{i+1} - v_i$. Then it is easy to check that the set $\{x_1, x_2, \ldots, x_k, x_1 + \cdots + x_k\}$ is monochromatic with respect to c. ■

Problem 3

Show that van der Waerden's theorem becomes false if we require that one of the two subsets contains *infinite* arithmetic progressions, by giving a counterexample.

Solution. Let $A = \bigcup_{k=1}^{\infty} [2^k - 1, 2^k - 1 + 2k - 1)$. A together with its complement defines a two-set partition of the positive integers. However, neither can contain an infinite arithmetic progression: Any two consecutive numbers in an AP with modulus r would differ by r, but A has 'gaps' larger than r (by choosing k such that $2^{k-1} > r$). Similar for the complement of A. This set also provides a counterexample to the Bonus problem (why?). ■

Bonus: Is it at least true that the finite arithmetic progressions of arbitrary length all start at the same number? That is, does it hold that whenever $\mathbb{N} = A_0 \cup A_1$, $A_0 \cap A_1 = \emptyset$, there exists a number m and an $i \in \{0, 1\}$ such that

$$\forall l \; \exists r \; \forall 0 \leq k \leq (l-1) \;\; m + kr \in A_i?$$

Justify!