Homework 1 for
MATH 497A, Introduction to Ramsey Theory

Due: Monday August 29

Problem 1

Use Ramsey's theorem for graphs to show that for every positive integer k there exists a number $N(k)$ such that if $a_1, a_2, \dots, a_{N(k)}$ is a sequence of $N(k)$ integers, it has a non-increasing subsequence of length k or a non-decreasing subsequence of length k. Show that $N(k+1) > k^2$.

Problem 2

Use Ramsey's theorem for graphs to show that for every positive integer k there exists a number $M(k)$ such that if the set $\{1, 2, \dots, M(k)\}$ is partitioned into two subsets, at least one of them contains a set of the form $\{x_1, x_2, \dots, x_k, x_1 + \cdots + x_k\}$.

(*Hint:* Consider a complete graph on vertices $\{0, 1, 2, \dots, M\}$, where M is an integer. Devise a 2-coloring of the graph so that a complete monochromatic subgraph on $k+1$ vertices yields the desired set.)

Problem 3

Show that van der Waerden's theorem becomes false if we require that one of the two subsets contains *infinite* arithmetic progressions, by giving a counterexample.

Bonus: Is it at least true that the finite arithmetic progressions of arbitrary length all start at the same number? That is, does it hold that whenever $\mathbb{N} = A_0 \cup A_1$, $A_0 \cap A_1 = \emptyset$, there exists a number m and an $i \in \{0, 1\}$ such that

$$\forall l \; \exists r \; \forall 0 \leq k \leq (l-1) \;\; m + kr \in A_i?$$

Justify!