Homework 9 for MATH 104
Due: Tuesday, November 21, 9:30am in class

Problem 1
(a) Suppose \( f : \mathbb{R} \to \mathbb{R} \) is differentiable at \( x \in \mathbb{R} \). Show that
\[
\lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} = f'(x).
\]

(b) Find an example of a function \( g : \mathbb{R} \to \mathbb{R} \) such that the limit in (*) exists for some \( x \in \mathbb{R} \) but \( g \) is not even continuous at \( x \).

Problem 2
Consider the functions
\[
f(x) = \sin\left(\frac{1}{x}\right) \quad g(x) = x \sin\left(\frac{1}{x}\right) \quad h(x) = x^2 \sin\left(\frac{1}{x}\right) \quad \text{for } x \neq 0,
\]
and set \( g(0) = h(0) = 0 \).

(a) Show that \( f \) cannot be extended continuously to \( x = 0 \), i.e. show that there is no continuous function \( \tilde{f} : \mathbb{R} \to \mathbb{R} \) such that \( \tilde{f}(x) = f(x) \) for all \( x \neq 0 \).

(b) Show that \( g \) is continuous but not differentiable at \( x = 0 \).

(c) Show that \( h \) is differentiable at \( x = 0 \) but \( h' \) is not continuous at \( x = 0 \).

Problem 3
(a) Use the mean value theorem to show that
\[
\sqrt{1 + x} < 1 + \frac{x}{2} \quad \text{for all } x > 0.
\]

(b) Suppose \( f \) that differentiable on \( \mathbb{R} \), that \( 1 \leq f'(x) \leq 2 \) for all \( x \in \mathbb{R} \) and that \( f(0) = 0 \). Show that \( x \leq f(x) \leq 2x \) for all \( x \geq 0 \).

Problem 4
Let \( f \) be differentiable on \( \mathbb{R} \) with \( a = \sup\{|f'(x)| : x \in \mathbb{R}\} < 1 \). Select \( s_0 \in \mathbb{R} \) and define \( s_n = f(s_{n-1}) \) for \( n \geq 1 \). Show that \( (s_n) \) converges.

[\text{Hint: Prove the inequality } |s_{n+1} - s_n| \leq a|s_n - s_{n-1}|.\]