Project1: Resonances and the asteroid belt

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Abstract

Asteroids in the solar system were simulated taking into account the gravitational forces from the Sun and Jupiter. Orbits and semi-major axes for asteroids with different periods were sketched versus time. Resonances were observed as in the histograms of number of asteroids with distances from the Sun at 3.275-3.300AU and 4.025-4.050AU. Also for those asteroids which have circular orbits, their distribution at a distance that is larger than 4.3AU is scarce. But for those asteroids whose speeds are slightly different from those of circular orbits, there is no obvious limit for their distribution.
I. Introduction

The number of asteroids versus their distances from the Sun shows some distinct gaps, called Kirkwood gaps.\textsuperscript{[1,2]} They were first discovered in 1857 by Daniel Kirkwood, who correctly explained that those gaps origin from the orbital resonances with Jupiter. For example, there are very few asteroids around 2.5\text{AU} from the Sun which is at 3:1 orbit resonance with Jupiter. Fig. (1) shows the histogram of number of asteroids against their distances from the Sun.\textsuperscript{[2]}

![Histogram of number of asteroids versus distances from the Sun.](image)

Fig. 1: Histogram of number of asteroids versus distances from the Sun. The ratios under arrows are the relationships between the periods of Jupiter and asteroids.

II. Methods

Rigorously, the problem of asteroids’ motion is a many-body motion problem. However, since the masses of the Sun and Jupiter are much greater than the other planets in the solar system, asteroids can be treated as if only the Sun and Jupiter were present. Moreover, the mass of the Sun is much greater than that of Jupiter which is much larger than those of asteroids, we can make such simplification that the Sun is stable and Jupiter
only feels the gravitational force from the Sun in the reference frame. Basically, I used ‘ODESolver’ and ‘RK45Multistep’ method to simulate the motion of an asteroid and ‘Draw’ to sketch its trace. More details are given in my programs.

In order to measure the semi-major axis of the asteroid’s orbit, a quantity \(-\frac{GMm}{2E}\) is introduced. It can be shown that this quantity equals the semi-major axis if the interaction of the asteroid with Jupiter is ignored. Here for parsimony, I derive this result for just circular orbits.

In a circular orbit, the gravitational force provides the centripetal force for this circular motion:

\[
\frac{GMm}{r^2} = m\frac{v^2}{r}
\]  

(1)

where \(r\) is the radius of this circle. The total energy of the asteroid is the sum of gravitational energy and kinetic energy:

\[
E = \frac{1}{2}mv^2 - \frac{GMm}{r}.
\]  

(2)

From Eq. (1) and Eq. (2) one can get:

\[
E = -\frac{GMm}{2r}.
\]  

(3)

Then it is straightforward from Eq. (3) that \(r = -\frac{GMm}{2E}\). Hence we can plot \(-\frac{GMm}{2E}\) versus time to know how the orbit of the asteroid changes with time.

III. Simulation and result

(a) Orbit of the asteroid at the 1/3 resonance

Initial state is set in the following way: the position of the asteroid is 2.52AU in the x-axis direction; the position of Jupiter is 5.24AU in the x-axis direction; their speeds are chosen so that they would move in a circle without the interaction between them. Fig. (2) shows the orbit of the asteroid at the 1/3 resonance after 109 years. The shape of the orbit is quite stable and I did not observe any obvious deviation from the original circle. Though I expected the deviation might be larger, the result from the simulation makes sense since the distance from the asteroid and Jupiter is relatively far and the influence from Jupiter is too small compared to that from the Sun.

(b) Orbits at 1/2, 3/7, 2/5, 2/3 resonances
Fig. (2), Fig. (3), Fig. (4) and Fig. (5) sketch the orbits at 1/2, 3/7, 2/5 and 2/3 resonances respectively. I can see the deviation of orbits in all of them, especially for 1/2 and 2/3 resonances. Their deviations are larger than that of 1/3 resonance. This is because of that the distances between the asteroid and Jupiter are shorter, as a result the gravitational forces from Jupiter are greater and the orbits are more influenced by the interaction of Jupiter. In the orbit at 1/2 resonance, wider band and narrow band of the orbit is observed. In another word, at some angle the deviation range is wider while at some other angle the deviation is narrower. This kind of different deviation range is not observed in the 2/3 resonance.

Fig. 2: Orbit at the 1/3 resonance.                          Fig. 3: Orbit at the 1/2 resonance.

Fig. 4: Orbit at the 3/7 resonance.                         Fig. 5: Orbit at the 2/5 resonance.

Fig. 6: Orbit at the 2/3 resonance.
(c) – \( \frac{GMm}{2E} \) versus time at 1/2, 3/7, 2/5 and 2/3 resonances

Fig. (7), Fig. (8), Fig. (9) and Fig. (10) show how the quantity – \( \frac{GMm}{2E} \) changes with time at the 1/2, 3/7, 2/5 and 2/3 resonances. We can see periodic fluctuations in the figures. After about 30 revolutions we can observe obvious fluctuations with larger period in the whole graph. The fluctuation with smaller period comes from the approaching and leaving Jupiter in one asteroid cycle while that with larger period comes from the ratio between the periods of the asteroid and Jupiter resulting the periodic changes of the whole configuration of the two. Fluctuations with larger periods are apparent for the 1/2 and 2/3 resonances and they have larger amplitude too. But they are not so obvious for the 3/7 and 2/5 resonances.

Fig. 7: – \( \frac{GMm}{2E} \) versus time at 1/2 resonance.

Fig. 8: – \( \frac{GMm}{2E} \) versus time at 3/7 resonance.
(d) Time dependence of \(-GMm/(2E)\) of different initial conditions

I set the initial position of the asteroid, \(x(1)\), from 2.0 to 5.0 in the x-axis direction in steps of 0.2 and look at the time dependence of the quantity \(-GMm/(2E)\) of the asteroid. When \(x(1)\) is small the distance between the asteroid and Jupiter is large, so the interaction between them is weak. I observed uniform oscillations in the graphs for \(x(1)\) from 2.0 to 3.0 with small amplitude. At 3.2 it begins to show the oscillation with larger period whose amplitude is small. At 3.4 the oscillation with larger period has larger amplitude, which becomes small again at 3.6 and vanishes at 3.8. After \(x(1)\) goes larger than or equal to 4.0, the oscillation with larger period appears in the graph for each of the initial positions. However, for \(x(1)\) at 4.6 the quantity has an unusual jump to a negative number less than 130 when time is about 163 years. I think the reason for this is that at that moment, the asteroid and Jupiter are very close to each other, and the speed of the asteroid is so large that the previously fixed ‘dt’ in my algorithm ‘RK45Multistep’ is not small enough. This breakdown of my algorithm was also observed when \(x(1)\) is 5.0.

Fig. (11), Fig. (12), Fig. (13), Fig. (14), Fig. (15), Fig. (16) and Fig. (17) show some typical shapes of the time dependence of \(-GMm/(2E)\). Fig. (11) and Fig. (12) are for
x(1) at 2.0 and 3.0, where there is almost no oscillation with large period. Fig. (13), Fig. (14) and Fig. (15) are for x(1) at 3.2, 3.4 and 3.6 respectively, among which the greatest amplitude of the larger period oscillation happens at 3.4. Fig. (16) sketches the graph of \(-GMm/(2E)\) when x(1) equals 4.6 where an unusual point is shown. Fig. (17) shows the graph for x(1) equals 5.0 which has many strange jumps. It shows that at 5.0 the asteroid and Jupiter becomes very close and my algorithm breaks down. In reality when the distance between the asteroid and Jupiter is small, the asteroid will be captured by the Jupiter and becomes Jupiter’s secondary planet or collides with Jupiter. Hence this kind of asteroid will not appear in the solar system for long.

![Fig. 11: –GMm/(2E) versus time when x(1) is 2.0.](image1)

![Fig. 12: –GMm/(2E) versus time when x(1) is 3.0.](image2)

![Fig. 13: –GMm/(2E) versus time when x(1) is 3.2.](image3)

![Fig. 14: –GMm/(2E) versus time when x(1) is 3.4.](image4)
(e) Histogram of the number of asteroids versus the value of $-\frac{GMm}{2E}$

I made a program to generate 151 asteroids with initial position $x(1)$ ranging from 2.0 to 5.0 in steps of 0.02 and get their values of $-\frac{GMm}{2E}$ at $t=2000$ and $t=5000$, as shown in Fig. (18) and Fig. (21) respectively. There is no major difference between histograms for $t=2000$ and $t=5000$. The only small difference I found is that there are less asteroids with semi-major axes larger than 6.0AU for $t=5000$. Unlike the histogram shown in Fig. (1), the ones I got do not have such sharp gaps where there are few asteroids. In my histograms one can just find some local minima which show that it is less probable to find asteroids at those distances.

I was not able to find the gap at 2.5AU. I increased the number of asteroids to 601 (see Fig. (19)) and shortened the bin to be 0.025 (see Fig. (20)) but still cannot find obvious gap at 2.5AU. I think it is because of at 2.5AU the interaction with Jupiter is so weak that
it can be neglected compared to that with the Sun. On the other hand, it correctly shows two obvious and reproducible minima at 3.3AU and 4.0AU which correspond to 1/2 and 2/3 resonances respectively. There are also some asteroids that have large \(-GMm/(2E)\) values after such long times. For those values which are larger than 10 the algorithm must broke down at some point, since from the conservation of energy, it is not possible for an asteroid to go as far as twice of its initial position. Moreover, by checking the warning information of eclipse when my program was running, I found that the algorithm broke down as soon as \(x(1)\) reaches 4.6. So those data after \(x(1)\) gets 4.6 are not suitable to analyze while those before 4.6 are reliable. Hence in my result, I showed 3.3AU and 4.0AU (3.275AU and 4.025AU more accurately) are two unusual distances where fewer asteroids are present. Moreover, my result shows that there is no asteroid if \(-GMm/(2E)\) is larger than 4.3AU, which sends me another useful message of the distribution of asteroids: their orbits are more likely to be inside the range of about 4.3AU.

![Fig. 18: Number of asteroids versus semi-major axis after 2000 years.](image-url)
Fig. 19: Number of asteroids versus semi-major axis after 2000 years for 601 asteroids.

Fig. 20: Histogram of number of asteroids versus semi-major axis with 601 asteroids and binning of 0.025 when $t=2000$. 

Fig. 20: Histogram of number of asteroids versus semi-major axis with 601 asteroids and binning of 0.025 when $t=2000$. 
(e) Situation where the initial velocity is deviated from that of a circular orbit

I used the same program as part (d) and got histograms for 151 asteroids and with binning step of 0.1. I tested for three different velocities which are 1%, 3% and 5% different from velocities of a circular orbit respectively. There histograms are shown in Fig. (22), Fig. (23), Fig. (24). The difference between them and those in part (d) is that there are more asteroid present in the range 4.3AU to 6.5AU, especially for 3% and 5% velocity differences. Fig. (25) shows the distribution of asteroids in the solar system.[3] From my result, those asteroids, whose speeds are different from those of circular orbits, are those brown and green ones in the picture.
Fig. 22: Histogram for 1% velocity difference.

Fig. 23: Histogram for 3% velocity difference.
Fig. 24: Histogram for 5% velocity difference.

Fig. 7: Asteroids in the solar system.
IV. Conclusion

I successfully obtained the gaps at 3.3AU and 4.0AU from my simulation and the
distribution limit for asteroids with circular orbits. However, at shorter distances there is
no gap in my simulation due to the weak interaction with Jupiter. For those asteroids
whose speeds are slightly different from those of circular orbits, I got that they do not
have obvious distribution limit which is apparent for asteroids with circular orbits. My
results basically match the real observation results and support the theory that Kirkwood
gaps origin from the orbital resonance with Jupiter.
V. References

