Mean-Motion Resonance and Formation of Kirkwood Gaps

Introduction

A histogram of the number of asteroids versus their distance from the Sun shows some distinct gaps, which are called Kirkwood gaps. The gaps are generally considered to be due to the mean-motion resonance in solar system. The gaps were first noticed in 1857 by Daniel Kirkwood, who also correctly explained their origin in the orbital resonances with Jupiter.

PHYS. Project 5.17 is dealing with this problem, and the purpose of the simulation is to show the formation of kirkwood gaps. If asteroids were in these gaps, their periods would be simple fractions of the period of Jupiter.

The resonance angle $\phi$ can be defined as:

$$\phi = (p + q) \lambda' - p \lambda - q \omega'$$

$$\lambda = n(t - \tau) + \omega$$

where $p$ and $q$ are integers.

As particle in Mean Motion Resonances with planet, the resonance angle of particle oscillates with some fixed value:

$$\frac{d\phi}{dt} = 0 \iff \frac{n}{n} = \frac{p + q}{p}$$

The theoretically most prominent gaps are located at semi-major axis of:

- 2.06 AU (4:1 resonance)
- 2.5 AU (3:1 resonance)
- 2.82 AU (5:2 resonance)
• 2.95 AU (7:3 resonance)
• 3.27 AU (2:1 resonance)

**Working Model**

The astroids have initially uniform surface density, and we assume the asteroid orbits’ initial position $x$ ranges from 1.9 to 4.4 in steps of 0.02. We use

$$v_y(i) = \sqrt{\frac{GM}{x(i)}}$$

to get the initial velocity in $y$ direction, so that the asteroids would have circular orbits if Jupiter were not present. We also assume that $y=0$ and $v_x=0$.

The Jupiter is initially with position $x=5.20$, which is generally in accordance with the solar system, and we also use the equation above to get its velocity in the $y$ direction. We set $M_{Jupiter} = 0.001M_{Sun}$ which is in accordance with the solar system.

Moreover, we assume the astroids are massless, so they have no gravitational force on Jupiter, which means Jupiter will have circular orbits.

We use $-\frac{GMm}{2E}$ to calculate the semi-major axis of the astroids in specific time, where $E$ is the asteroid kinetic energy plus the asteroid-Sun potential energy.

**Results:**

1. Orbital evolution of astroids

For most asteroid orbits, the perturbation of Jupiter can lead to small oscillation of the semi-major axis. The farther away the asteroid is from Jupiter, the smaller the amplitude of the oscillation is. The larger semi-major axis the asteroid has, the longer period of the oscillation has. (See Figure 1 & Figure 2)
However, for asteroids whose periods are simple fractions of the period of Jupiter, the evolution would be different. (See Figure 3)
From Figure 3, we find that the semi-major axis of the asteroid decreases during the oscillation and then increases until the original semi-major axis. We find the asteroid at the 3:2 resonance point also has similar performance. In that case, asteroids near these resonance points can deviate from its original semi-major axis for a long time, which will cause the formation of Kirkwood Gaps.

However, if we choose the 3:1, 4:1, 5:2, 7:3 resonance point or so, there is no obvious feature as above. The possible reason is that if the integers in the ratio is larger, the resonance effects get weaker. And the 3:1 and 4:1 resonance points are too close to the sun, so the orbit of the asteroid will get more stable. I tried to modify the mass of the planets to 50

Figure 3: The orbital evolution of asteroid at 3.28 AU, which is the 2:1 mean motion resonance.
$M_{\text{Jup}}$, but the resonance effects are still not obvious.

(2) Histogram of the number of asteroids

From Figure 4 we can see that two gaps formed during the 2000 years, and these gaps are related to 2:1 and 3:2 resonance points.

![Figure 4](image)

**Figure 4**: Distribution of semi-major axis of asteroids after 2000 years.

Actually, the gaps don't always exist in the 2000 years, and sometimes the gaps disappear. But most of the time we can see the gaps in the diagram. So the resonance is a possible cause of Kirkwood gaps.

![Figure 5](image)

**Figure 5**: The histogram of the number of planets at $t=2000$
The gaps can also be identified in the histogram, it is obvious that the density in the gaps are much smaller than average.

If we simulate for longer time, we get similar results. (See Figure 6 & Figure 7)

Figure 6: Distribution of number of asteroids after 5000 years.

Figure 7: Histogram of number of asteroids at 5000 years.

In the 5000 years case, I reduced the number of asteroids, because if the asteroid is too close to Jupiter, it would cause some problems when running the program. The most common problem is the ODESolver(RK45MultiStep) does not converge, which may affect the accuracy of the simulation.
Actually, I found that most astroids near Jupiter will be scattered many times during the 2000 or 5000 years in the simulation.

To check the result, in another simulation, I set the phase between Jupiter and astroids to be 180 degree when the simulation starts, and I got similar results after 2000 years. (See Figure 8)

![Figure 8: The histogram of numbers of astroids. The astroids and Jupiter have initial difference in phase of 180 degree.](image)

(3) Effects of Eccentricity

Next we discuss the effects of the eccentricity. In the simulation showed by Figure 9, we set the astroids to have initial velocities that vary from their values for a circular orbit by 1%. We find that the result is quite similar the the circular orbit case. Although in Figure 9 you can't see the gap at 3:2 resonance point, actually during the simulation, the gap does appear. We can also see that some astroids are scattered to the outer area by Jupiter.
If we change the variation from 1% to 5%, we have the results shown by Figure 10 and Figure 11. We see the gaps still exists, but more asteroids are scattered out by Jupiter.

Figure 9: The histogram of number of asteroids when the initial velocity of asteroids vary 1% from circular velocity.

Figure 10: The histogram of number of asteroids when the initial velocity of asteroids vary 5% from circular velocity.
Finally we will see what happens if the velocity of Jupiter is 5% larger than the circular velocity. Actually, the orbit of Jupiter is not circle, it has an eccentricity of about 0.05.

From Figure 12, we see that even Jupiter has eccentricity, it does not affect the formation of the gaps, and we can still see the features in Figure 12.
Conclusion:

From the results above, we confirm that mean-motion resonance can be a cause of the formation of Kirkwood gaps. This process is generally stable, as we show above, it is not affected much by the eccentricity of the asteroids and Jupiter.

The most prominent gaps in the simulation are 2:1 and 3:2 resonance point. The failure of observation of other gaps may result from the small width, or the long time scale needed. The accuracy of the simulation also affected the result, and limited the range of the asteroids in the simulation. If we still want to use the RK45MultiStep ODESolver, we need to change the maximum number of iteration steps in the class. Otherwise, we need to improve the method so that it can converge when the two objects get close to each other.

Figure 12: The histogram of number of asteroids, when Jupiter has eccentricity.