Kirkwood Gaps in Asteroid Belt

Project #1 (Project 5.18 in text) for PHYS 527, 2008 Fall

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The density function of asteroid distribution in our solar system has certain “diffuse” points at some special distances from the Sun. These “diffuse” points, where it is more unlikely to observe asteroids, are called Kirkwood Gaps, named after American astronomer Daniel Kirkwood who discovered and successfully explained this phenomenon.

These gaps are formed due to resonance, more specifically, due to the resonance with Jupiter’s period. Gaps occur at the orbits where the orbital periods are simple fractions of Jupiter’s. The mainly observed gaps are shown in the histogram below.

(This histogram is from Wikipedia, under “Kirkwood Gaps”, and it is originally given by NASA.)
a. Orbit for 1/3 Resonance

Related Classes: (Under Project1.Xuesong) Planet2.java, Planet2App.java

Remarkable Changes:
Planet2.GM1 = 0.001*GM, approx. mass of Jupiter;
Adding semi-axis drawing to Planet2App.java.

The default value given in the original program, namely, x1 = 2.52 (AU), corresponds to the 1/3 resonance. Orbit is shown in Figure 1.

Except for some small wobbling, the orbit is generally steady, in terms that the asteroid has relatively steady semi-axis.

b. 4 More Resonant Orbits

Related Class:
Planet2.java, Planet2App.java

Remarkable Changes:
Set initial conditions for other different resonances in control frame of Planet2App.java.

Consider the orbit with period aT, where a is a small fractional number, and T is the period of Jupiter. According to Kepler’s 3rd Law, it gives,

\[ r = \sqrt[3]{\frac{a^2T^2}{T^2}R^3} = \sqrt[3]{a^2R^3} \]

where R is the semi-diameter of Jupiter, and r is that of the asteroid with period aT.

Thus for 1/2, 3/7, 2/5, 2/3 fractional period, the corresponding semi-diameters of those four asteroids are:

\[ x1 = 3.301, 2.979, 2.845, 3.999 \text{ (AU)}. \]
For initial speed $v_y1$, by applying Newton’s law for a circular orbit around central mass $M$, 

$$\frac{GMm}{r^2} = \frac{m(vy1)^2}{r}$$

which is, 

$$vy1 = \sqrt{\frac{GM}{r}}.$$

Set initial values of $x_1$, $vy1$ for those five asteroids (including the 1/3 resonance one in part a) respectively, and then plot the orbits for a few periods. Graphs are attached at the end of this report.

The orbits of 2/5, 3/7 resonance asteroids are more steady than 1/2 and 2/3, while the latter two are closer to Jupiter, with 2/3 the nearest and wobbling the most.

c. Semi-axes Changing with Time – Five Asteroids

Related Classes:

- Asteroids.java – Class of N asteroids. OED Implemented, using RK45MultiStep. N is set to 5.
- AsteroidPlot.java – Class for drawing semi-axis for each asteroid in one frame. Extension of AbstractSimulation.

For circular orbit, the speed satisfies, 

$$v = \sqrt{\frac{GM}{r}}$$

as derived above. Thus the energy $E$ can be rewritten into, 

$$E = \frac{1}{2}m\left(\frac{GM}{r}\right) - \frac{GMm}{r} = -\frac{GMm}{2r}$$

That is, 

$$r = -\frac{GMm}{E}$$

By adding methods for calculating energy $E$ and then semi-axis $r$, AsteroidPlot.java gives out result for the changing of semi-axis with respect to the time $t$. The following graph, Figure 2, is for the above five resonance conditions, with initial velocity set to achieve a circular orbit if Jupiter is absent (just as before).
Again consistent with part b, 1/2 and 2/3 have semi-axes which change more dramatically, as, again, they are closer to Jupiter.

**Figure 2, Semi-axes changing with time, x-axis as time, in years, y-axis as semi-axes, in AU.** From below to above, the curves represent for 1/3, 2/5, 3/7, 1/2 and 2/3 resonance semi-axes. Upper time limit was chosen as 300 years, which is about 30 revolutions for 2/3 resonance (the longest). Due to System limit, dt (step) in ODE is set to be 0.1, but results are consistent with the one of smaller dt (not capable of recording up to 300 years).

**d. Fifteen Asteroids’ Semi-axes Changing**

Related Classes:
**Asteroids.java, AsteroidPlot.java,**

Remarkable Changes:
N in Asteroids.java is set to be 15. Codes in AsteroidPlot.jave are modified for plotting the motions of N asteroids’ instead of only 5.
Initial position $x_1$ is set to be from 2.0 to 5.0, with 0.2 as step, and thus generating 15 asteroids. Initial velocity (value $v_{y1}$) is set as usual, for each asteroid, to satisfy “circular conditions” derived above. Semi-axes are calculated using the same equation as in part c. Results are shown in Figure 3.

As demonstrated in Figure 3, closer (to Jupiter) asteroids have semi-axes which change more dramatically. Note that asteroids with semi-axes which were initially beyond 4.4 AU fall back into <4.4 AU region, resulting in a “empty space” from time to time beyond 4.4. Those are the most unusual ones.
e&f. Histogram of Asteroid Distribution (& Some Discussion)

Related Classes:
- **Asteroids.java**
- **AsteroidCalculation.java** - Class for calculating semi-axes after $t = 2000$ or $5000$ years, including a frame giving histogram of the asteroid semi-axis number distribution after calculation.

Remarkable Changes:
N in Asteroids.java is set to be 150 or 125, subject to different conditions.

Initial set of semi-axes $a[i]$ is given the value from 2.0 to 4.5, with a 0.02 step, speeds are given as circular speeds as usual, or with 1%, 2%, or 5% deviation. By doing similar calculations as above (just let the ODEsolver advance for 2000 or 3000 steps and use the final results for states), we could have a final set of semi-axes $a[i]$, where $i=0, 1, 2...149$ or 124. Binning those values with a bin width 0.1, Figure 4 to Figure 9 are obtained. Some simple observation and discussion are written below each figure.

Here only the N=125 (from x1=2.0 to x1=4.5, with 0.02 step) results are given, because x1 values beyond will yield divergence in RK45MultiStep, and later cause system halt. Some of the results given were involved with RK45MultiStep convergence problems as well, but at least they (and the system) managed to generate the final results, which are some of the histograms shown.

Why asteroids with initial semi-axes (denoted as “a” here and after) larger than 4.5 AU are easy to cause divergence? Is it reasonal to eliminate them? As shown in Figure 3, the asteroid with initial a which is beyond 4.4 tend to have very unstable orbits, and fall back into inner regions. Farther asteroids, which are too closer to Jupiter, technically cause the RK45MultiStep convergence problem, while physically, they are also likely to be captured by Jupiter and become its satellite finnally (as can be (almost) seen in the animation in Planet2App.java). This is almost the real case in astronomy, as the outer edge of the Asteroid Belt lies around 3.5 – 4 AU, with really few asteroids beyond 4.2 AU, mainly due to the influence of Jupiter.

One method that I used but not mentioned in the text book is the so called “accumulated” distribution (shown in Figure 10). It works in the following way: as time advances, e.g. AsteroidPlot.java makes one step, the histogram records the occurrences of asteroids within 0.1 AU bin, and accumulates them as time goes by. This is, in some way, showing the distribution of asteroids as well, just like counting the times of their appearance at some certain spot, and determining which spot is their favorite and which is not liked so much.
Figure 4. Distribution Histogram of Asteroid Semi-axes, x-axis as different length of semi-axis, and y-axis the number of asteroid in 0.1 neighborhoods. Initial speeds are given the same as circular orbital speeds. Results are those 2000 years after initialization. Kirkwood gap at a around (\( \sim \)) 3.3 AU can be seen clearly, while the one at a \( \sim 4 \) AU is just relatively less dense to its neighbor, but already reaches the average value actually. Thus technically, only gap at a \( \sim 3 \) AU comes up in the simulation. There are two peculiar value of a beyond 7 – such values are cut off in the following figures to give a better view of the bulk distribution.
Figure 5, Distribution Histogram of Asteroid Semi-axes, x-axis as different length of semi-axis, and y-axis the number of asteroid in 0.1 neighborhoods. Initial speeds are given the same as circular orbital speeds. Results are those 5000 years after initialization. Kirkwood gap at a around (~) 3.3 AU again can be seen clearly, together with a much better observation of the one at ~ 4 AU. Technically, the 5000-year long simulation gives a better result to some extent, which is as expected.
Figure 6, Distribution Histogram of Asteroid Semi-axes, x-axis as different length of semi-axis, and y-axis the number of asteroid in 0.1 neighborhoods. Initial speeds are given the same as circular orbital speeds with 5% deviation. Results are those 2000 years after initialization. Kirkwood gap at around (~) 3.3 AU again is emerged, with a even better observation of the one at a ~ 4 AU. Gaps seem to occur more easily if orbits have initial eccentricity. Extinction beyond 4.2 AU is (slightly) there as well.
Figure 7, Distribution Histogram of Asteroid Semi-axes, x-axis as different length of semi-axis, and y-axis the number of asteroid in 0.1 neighborhoods. Initial speeds are given the same as circular orbital speeds with 2% deviation. Results are those 2000 years after initialization. Kirkwood gaps at ~ 3.3 AU and the one ~ 4 AU are there, but it seems that other gaps begin to emerge as we have lower spots almost the same position with the last one (for example, at a = 2.9 AU). However, due to lack of samples, as the deviation is still within Poisson, it is difficult to say whether this is random or real gaps. Extinction beyond 4.2 AU looks just as obvious.
Figure 8, Distribution Histogram of Asteroid Semi-axes, x-axis as different length of semi-axis, and y-axis the number of asteroid in 0.1 neighborhoods. Initial speeds are given the same as circular orbital speeds with only 1% deviation. Results are those 2000 years after initialization. Kirkwood gaps at ~ 3.3 AU and the one ~ 4 AU are there again. Less eccentricity seems to produce fewer gaps (as the density at lower values are almost even). Extinction beyond 4.2 AU is there as well.
Figure 9, Distribution Histogram of Asteroid Semi-axes, x-axis as different length of semi-axis, and y-axis the number of asteroid in 0.1 neighborhoods. Initial speeds are given the same as circular orbital speeds with only 1% deviation again. Results are those 5000 years after initialization. Kirkwood gaps at ~ 3.3 AU and the one ~ 4 AU are there again. This is actually the only one which was not affected too badly by the convergence problem of RK45MultiStep. Other deviation percentages, i.e. 2% and 5%, cannot be tested for a 5000-year time as the errors accumulate and system halts. It does not look to much different from Figure 8, which is with 1% deviation as well but only 2000-year after initialization.
Figure 10, “Accumulated” Distribution Histogram of Asteroid Semi-axes, x-axis as different length of semi-axis, and y-axis the number of occurrences of an asteroid in 0.1 neighborhoods. Initial speeds are given the same as circular orbital speeds with only 1% deviation. This gives a somehow clearer view of the distribution, in terms that it counts the “favor” of those asteroids, i.e. checking which places they are most likely to stay. Gaps ~3.3 AU and 4 AU are obvious, and extinction beyond 4.2 AU is also there.
Appendix for part b, 4 orbits

Figure 11, 1/2 Resonance Orbit
Figure 12, 2/3 Resonance Orbit
Figure 13, 2/5 Resonance Orbit
Figure 14, 3/7 Resonance Orbit