Modeling Realistic Projectile Motion

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1. Introduction

The calculation of simple projectile motion is one of the most basic exercises in physics. Finding, for example, the range of a thrown object, is often one of the first problems attempted in introductory physics classes. It is always stressed that these calculations are done in the absence of friction; it is much less commonly pointed out that the introduction of even a simple frictional force renders analytical solutions impossible.

Fundamentally, drag results from myriad collisions between the thrown object and the particles in the atmosphere. Direct modeling of this process is impossible, due to the sheer number of such collisions on any real-world scale. An empirical approximation of the drag force was derived by John Strutt (aka Lord Raleigh):

\[ F_{\text{drag}} = -\frac{1}{2} \rho v^2 A C_d \]  \hspace{1cm} (1)

Here, \( \rho \) is the atmospheric density, \( v \) is the velocity, \( A \) is the cross-sectional area of the object, and \( C_d \) is an empirical coefficient of drag, which is unique to individual objects. This force, as would be expected, acts opposite the direction of motion.

Just including the basic drag force still does not render a realistic trajectory. It is much more accurate than before, but in the real world there are also effects from rotational motion. Specifically, as a ball rotates the instantaneous velocities at different points along the ball are different. For example, a ball with backspin will have a higher relative velocity at the bottom of the ball than the top. Thus the force at the bottom of the ball is greater producing an upward net force. This process can be modeled by:

\[ F_{\text{Magnus}} \sim v \Delta v \]  \hspace{1cm} (2)

This can be expressed as a function of the angular velocity of the ball thusly:

\[ F_{\text{Magnus}} \sim vr\omega \]  \hspace{1cm} (3)

As previously noted, pure backspin should result in a lift force perpendicular to the velocity of the ball. This implies that the empirical expression of the Magnus force should be:

\[ F_{\text{Magnus}} = mC_m(\omega \times v) \]  \hspace{1cm} (4)
Here, $m$ is the mass of the ball, and $C_m$ is an experimental coefficient encompassing factors similar to those of the drag coefficient, $C_d$.

I have attempted to write a java class to model the behavior of a projectile under these influences. Due to time constraints, I have limited the program to modeling the motion in two dimensions—considering only pure backspin or topspin. As such, in combining the equations above, the projectile’s motion is expressed by the following:

$$\frac{dv_x}{dt} = -C_d |v_x| v_x + C_m (-\omega_z v_y) \quad (5)$$

$$\frac{dv_y}{dt} = -C_d |v_y| v_y + C_m (\omega_z v_x) \quad (6)$$

It is interesting to note that the nature of the Magnus force results in a coupling of the $x$ and $y$ velocities. It is easy to see why there exists no analytic solution. For the purposes of this model, I have employed a Runge-Kutta algorithm for solving these equations numerically. The program is designed to plot the trajectory of the ball in the $x$-$y$ plane, and simultaneously plot the $x$ and $y$ positions as a function of time. The controller is set up so that the user may set values for the mass, drag and Magnus coefficients, the initial velocity, the initial angle (in radians), and the angular velocity (in rad/s). The values chosen as defaults represent an average, well-thrown baseball, though the Magnus coefficient is set slightly higher, to more easily see the effects of the angular velocity.

### 2. Running the Model

Prior to running the program, it is important to discuss what is expected. First, with the Magnus and drag coefficients set to zero, the program should return a basic parabolic trajectory. This is shown in Figure 1. Next, we expect that the trajectory will be skewed right by the drag force alone (i.e. the $x$-velocity will decrease continuously, resulting in a non-parabolic shape). This expectation is also confirmed in 2. The most interesting part of the model, though, is the effect of spin. If we apply a modest angular velocity of 10 rad/s, we can see a noticeable difference from the situation with only drag. Indeed, the backspin has increased the range of the projectile by a small amount (see Figure 3). Backspin similarly yields an expected result of decreased range.

A more interesting question is the effect of such interactions on the maximum range of a projectile. The maximum range in the absence of drag is 45 degrees, and is very easy to compute. But the Magnus force could change that. At relatively higher angles, the Magnus force will act to decrease the forward ($x$) velocity relatively more. As the launch angle decreases, the Magnus force points more and more vertically, which increases the time in the
air beyond what it would be otherwise. This could result in the maximum distance being reached at lower angles for backspin, and higher angles for topspin. The remaining figures test this hypothesis by using default values, except for the angular velocity, and angle. The launch angle is ranged in 2 degree increments from 45 degrees to 23 degrees, and the angular velocity is set to 10 rad/s. This value was chosen because it allows the changes in range to be visible, and is not high enough to cause problems (see below). The results are very interesting. For these specific values, the introduction of spin effects decreases the optimal launch angle to approximately 31 degrees. This is somewhat surprising. A relatively low rotational speed apparently has a large effect on the flight of the ball. It should be noted that this optimal angle will change with changes in the Magnus force or to the other variables. Once we include these complexities, it becomes obvious that these sorts of answers are variable.

3. Model Validity

As mentioned previously, there is no analytical solution to the problem of projectile motion in the presence of drag forces. This makes determining the validity of the model difficult. Indeed, it is not even possible to check for conserved quantities, because this is a non-conservative system (unless we were to model each individual interaction, which is obviously impossible). Simpler checks can be made, however. First, if we set the drag coefficient, and the Magnus coefficient to zero, we should get back the standard parabolic trajectory. As shown above, this is indeed the case.

4. Bugs and Failures

Through testing of the program, one notable failure was discovered. At sufficiently high values for the spin (relative to the mass and the drag coefficients and velocity), the model obviously breaks down. Specifically, after the projectile starts, it rises due to the lift from the Magnus forces (assuming backspin here), then continues to curve upward until it actually gains a negative velocity. Odd loops and other strange behaviors follow. It appears that this situation is not realistic. However, the effect of the Magnus force, as described above, is an acceleration perpendicular to the direction of velocity. So, as the ball rises, and friction acts, it will eventually be traveling upward with no x-velocity. But due to the nature of the program, the Magnus force will still be acting, causing the object to gain a negative velocity, and hence, change directions.
The difficulty in determining whether this is actually a problem with the program is partly due to the fact that such a situation does not arise often in reality. Objects light enough to gain such a high rate of rotation quickly stop rotating so fast precisely because of friction. The program assumes that the magnus force remains constant throughout flight, which is obviously not the case. Indeed, by assuming that the angular velocity is constant, the program is essentially modeling some sort of powered flight at high values. If this is the case the results are not necessarily absurd.
Fig. 1.— Projectile motion at 45 degree launch angle, with no drag or spin
Fig. 2.— Same as Figure 1, but now including a drag force. Note the change in scale
Fig. 3.— $\omega = 10$ rad/s. Launch angle 45 degrees
Fig. 4.— Same as previous, now with $\omega = -10 \text{ rad/s}$
Fig. 5.— We begin a series of graphs showing the change in maximum range due to backspin. The following graphs all have $\omega = 10$ rad/s, initial velocity of 40 m/s, and the default mass, $C_d$ and $C_m$. This first graph shows the launch angle at 42 degrees.
Fig. 6.— 40 degrees
Fig. 7.— 37 degrees
Fig. 8.—35 degrees
Fig. 9.— 33 degrees
Fig. 10.— 31 degrees
Fig. 11.— 29 degrees
Fig. 12.— 26 degrees
Fig. 13. — 23 degrees