Sequences (Part 3)  
*Supplemental Material Not Found in You Text*

Motivating Examples

- Geometric Sequences will help us answer the following:

- An interest-free loan of $12,000 requires monthly payments of 15% of the unpaid balance. What is the unpaid or outstanding balance after 18 payments?

- Suppose a business makes a $1,000 profit in its first month and has its monthly profit increase by 10% each month for the next 2 years. How much profit will the business earn in its 24\textsuperscript{th} month? How much profit total profit will the business have earned at the end of 2 years?

Geometric Sequences

- A **Geometric Sequence** is a sequence where the *ratio* between any two consecutive numbers in the sequence is a constant.

\[ a_{k+1}/a_k = r \text{ where } r \text{ is a constant} \]

- Examples of Geometric Sequences:
  
  - \(1, 4, 16, 64, \ldots\) (has common ratio \(r = 4\))
  
  - \(32, 16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots\) (has common ratio \(r = \frac{1}{2}\))

Finding a Formula for a Geometric Sequence

- Consider the Geometric Sequence: \(1, 4, 16, 64, \ldots\)

<table>
<thead>
<tr>
<th>Index (Order)</th>
<th>Sequence Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1 = 1 = 1 = 1(4)^0</td>
</tr>
<tr>
<td>1</td>
<td>4 = 1(4) = 1(4) = 1(4)^1</td>
</tr>
<tr>
<td>2</td>
<td>16 = 4(4) = 1(4)(4) = 1(4)^2</td>
</tr>
<tr>
<td>3</td>
<td>62 = 16(4) = 1(4)(4)(4) = 1(4)^3</td>
</tr>
</tbody>
</table>

So we can write a formula for the \(n\)\textsuperscript{th} term:

\[ a_n = 1(r)^n \text{ where the index starts with } n = 0 \]
The Formula for a Geometric Sequence

- A geometric sequence can be written as

\[ a_0, \ a_1, \ a_2, \ a_3, \ a_4, \ \ldots \]
\[ a_0, \ a_0 \cdot r, \ a_0 \cdot r^2, \ a_0 \cdot r^3, \ a_0 \cdot r^4, \ \ldots \]

Where \( a_0 \) is the first term of the sequence and \( r \) is the common ratio

- The \((n+1)\)\textsuperscript{st} term of a geometric sequence is (index starts at \( n=0 \))

\[ a_n = a_0 r^n \]

- It can also be useful to remember that each term is \( r \) times the previous term.

\[ a_k = a_{k-1} \cdot r \]

Geometric Sequence Example:

- For the Geometric Sequence 9, 12, 16, 21\( \frac{1}{3} \), . . .

1. What is the common ratio?
2. What is the first term in the sequence?
3. What is the fifth term in the sequence?
4. Find a formula for \( a_n \), (the \( n+1 \)\textsuperscript{st} term in the sequence)
5. What is the 9\textsuperscript{th} term in the sequence? (Round to 4 decimal places)
6. What is the 20\textsuperscript{th} term in the sequence? (Round to 4 decimal places)
Geometric Sequences and Compound Interest

- Recall that the compound interest formula is:

\[ FV = PV(1 + i)^n \]

- Geometric Interest Formula:

\[ a_n = a_0(r)^n \]

- Compound Interest is a Geometric Sequence:

The first term: \( a_0 = PV \)

The common ratio: \( r = (1 + i) \)

Revisiting Our First Compound Interest Example:

- (From 3.1 Notes) Suppose Olaf invests $5,000 in an investment that pays 6% interest compounded annually. How much does he have at the end of each of the first 5 years?

- (From 3.1 Notes)

<table>
<thead>
<tr>
<th>Year</th>
<th>Interest earned that year</th>
<th>Balance at end of year</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$5000 \times 0.06 \times 1 = $300</td>
<td>$5300.00</td>
</tr>
<tr>
<td>2</td>
<td>$5300 \times 0.06 \times 1 = $318</td>
<td>$5618.00</td>
</tr>
<tr>
<td>3</td>
<td>$5618 \times 0.06 \times 1 = $337.08</td>
<td>$5955.08</td>
</tr>
<tr>
<td>4</td>
<td>$5955.08 \times 0.06 \times 1 = $357.30</td>
<td>$6312.38</td>
</tr>
<tr>
<td>5</td>
<td>$6312.38 \times 0.06 \times 1 = $378.74</td>
<td>$6691.12</td>
</tr>
</tbody>
</table>

- So 5000, 5300, 5618, 5955.08, 6312.38, 6691.12, ... is a geometric sequence,

- with \( a_0 = $5300 \)
- and \( r = 1.06 \)

\[ r = \frac{\$5300}{\$5000} = 1.06 \]
Real World Examples 1

- An interest-free loan of $12,000 requires monthly payments of 15% of the unpaid balance. What is the unpaid or outstanding balance after 18 payments?

- Your starting balance (after 0 months) is 12,000

- After 1 month, you owe a payment:
  Your first payment is 15% of $12,000
  $0.15 \times 12000 = 1800$
  Remaining Balance (after 1 month) $12,000 - 1800 = 10,200$

- Your second payment is 15% of $12,000
  $0.15 \times 10,200 = 1530.$
  Remaining balance (after 2 months) is $10200 - 1530 = 8670$

- So this is a geometric sequence:
  $- \frac{a_1}{a_0} = \frac{10,200}{12,000} = 0.85$
  $- \frac{a_2}{a_1} = \frac{8670}{10200} = 0.85$

- Since $a_1$ is balance remaining after month 1, and $a_2$ is balance remaining after month 2 ....

- Balance remaining after 18 months is....
  $a_{18} =$

\[
\begin{array}{|c|c|}
\hline
\text{Month(s)} & \text{Unpaid Balance} \\
\hline
0 & \$12,000 \\
1 & \$10,200 \\
2 & \$8670 \\
\hline
\end{array}
\]
Finding Formula for the Sum of the First $n$ Terms of a Geometric Sequence:

- Find the $n^{th}$ partial sum of the geometric series

\[
\begin{align*}
S_n &= a_0 + a_1 + a_2 + \ldots + a_{n-1} \\
S_n &= a_0 + a_0r + a_0(r)^2 + \ldots + a_0r^{n-1}
\end{align*}
\]

- Now for a clever trick

\[
\begin{align*}
S_n &= a_0 + a_0r + a_0(r)^2 + \ldots + a_0r^{n-1} \\
r(S_n) &= r\left(a_0 + a_0r + a_0(r)^2 + \ldots + a_0r^{n-1}\right) \\
r(S_n) &= a_0r + a_0rr + a_0(r)^2r + \ldots + a_0r^{n-1}r \\
r(S_n) &= a_0r + a_0(r)^2 + a_0(r)^3 + \ldots + a_0r^n
\end{align*}
\]

- Subtract the 2 Equations

\[
\begin{align*}
S_n &= a_0 + a_0r + a_0(r)^2 + \ldots + a_0r^{n-1} \\
-r(S_n) &= -\left(a_0r + a_0(r)^2 + a_0(r)^3 + \ldots + a_0r^n\right)
\end{align*}
\]

\[
S_n - rS_n = a_0 - a_0r^n
\]

- Now a bit of Algebra

\[
\begin{align*}
S_n - rS_n &= a_0 - a_0r^n \\
S_n(1 - r) &= a_0(1 - r^n) \\
\frac{S_n(1 - r)}{(1 - r)} &= \frac{a_0(1 - r^n)}{1 - r} \\
S_n &= \frac{a_0(1 - r^n)}{1 - r}
\end{align*}
\]

Formula for Partial Sum of Geometric Sequence

- The sum of the first $n$ terms of a geometric sequence with first term $a_0$ and common ratio $r$ is

\[
S_n = \frac{a_0(1 - r^n)}{1 - r}
\]

As long as $r$ is not equal to 1.
Partial Sum of Geometric Sequence Example:

• Find the 20\textsuperscript{th} partial sum of the Geometric Sequence 1000, 1200, 1440, 1728, \ldots
  
  – Identify $a_0$

  – Identify $r$

  – Plug into $S_n$ formula.

Real World Examples 2

• Suppose a business makes a $1,000 profit in its first month and has its monthly profit increase by 10\% each month for the next 2 years. How much profit will the business earn in its 24\textsuperscript{th} month? How much profit total profit will the business have earned at the end of 2 years?

  – Work out a few months Remember, we need the index to start at 0.

<table>
<thead>
<tr>
<th>Month</th>
<th>Index</th>
<th>Monthly Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Month 1</td>
<td>0</td>
<td>$1,000</td>
</tr>
<tr>
<td>Month 2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Month 3</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

  – Realize this is a Geometric Series
    * Identify $a_0$
    * Identify $r$

  – How much profit will the business earn in its 24\textsuperscript{th} month?

  – How much profit total profit will the business have earned at the end of 2 years?
Shortcut to Finding $r$

• Sometimes there is an easier way to find $r$ rather than working out several terms and checking the ratio.

• If each term in the sequence is a certain percent more than the previous term,$$
  r = 1 + p
$$
  (where $p$ is the percent, converted to a decimal)

• If each term in the sequence is a certain percent less than the previous term,$$
  r = 1 - p
$$
  (where $p$ is the percent, converted to a decimal)

Real World Examples 3

1. You start advertising your dog grooming business on a new social network called Woofer. Your advertising cost in January $100. Since the social network is growing in popularity, your advertising cost in February are 8% higher. You assume this pattern will continue, and each month your advertising cost will be 8% higher than the previous month.

   (a) What are your advertising costs in December?
   (b) How much will you spend in advertising over the year?

2. A business has a profit of $25,000 in the first year and then loses 5% each year for the next seven years.

   (a) What is the business’ profit in the 4th year?
   (b) What is the total profit after 7 years?
Homework

1. Write a formula for the following geometric sequences. Assume the sequences all start with index \( n = 0 \).
   
   (a) 5, 15, 45, 135, \ldots  
   
   (b) 972, 324, 108, 36, \ldots  
   
   (c) 5000, 6200, 7688, 9533.12, \ldots  
   
   (d) 10000, 9150, 8372.25, 7660.60875, \ldots  

2. Find the indicated term of the following geometric sequences: (round your final answer to 2 decimal places)
   
   (a) Find the 12\(^{th}\) term of 5, 15, 45, 135, \ldots  
   
   (b) Find the 8\(^{th}\) term of 972, 324, 108, 36, \ldots  
   
   (c) Find the 10\(^{th}\) term of 5000, 6200, 7688, 9533.12, \ldots  
   
   (d) Find the 6\(^{th}\) term of 10000, 9150, 8372.25, 7660.60875, \ldots  

3. Find the indicated partial sum of the given geometric sequence: (round your final answer to 2 decimal places)
   
   (a) Find the 12\(^{th}\) partial sum of 5, 15, 45, 135, \ldots  
   
   (b) Find the 8\(^{th}\) partial sum of 972, 324, 108, 36, \ldots  
   
   (c) Find the 10\(^{th}\) partial sum of 5000, 6200, 7688, 9533.12, \ldots  
   
   (d) Find the 6\(^{th}\) partial sum of 10000, 9150, 8372.25, 7660.60875, \ldots  

4. A business made a profit of $8000 in their first year. They expect their yearly profit to increase by 7.4\%. What will their profit be in their 10\(^{th}\) year?

5. A business made a profit of $8000 in their first year. They expect their yearly profit to increase by 7.4\%. How much profit will they make in their first 10 years?

6. A business spent $50,000 on their raw materials during April. Due to a recession they expect their monthly raw material cost to decrease by 1.5\% each month. How much did they spend on raw materials for the 10 month period of April - January?
7. A business spent $50,000 on their raw materials during April. Due to a recession they expect their monthly raw material cost to decrease by 1.5% each month. How much did they spend on raw materials in January?

8. Sadly, Teddy is addicted to drugs. This week he spent $10 on his drug habit. Next week he will spend 4% more on drugs. And, since drugs are so addictive, we assume this pattern will continue, that he will spend 4% more on drugs each week than he spent the previous week.

   (a) How much will Teddy spend on drugs during the second week?
   (b) How much will Teddy spend on drugs during the last week of the year?
   (c) How much did Teddy spend on drugs over the course of this year?

9. A food company normally has weekly sales of $200,000. However, a fad diet that recommends not eating their product is gaining in popularity. They expect their weekly sales to decrease by 8% each week as more and more people participate in this fad diet. What will their profit be the 4th week after this fad diet took hold? (Assume we’re counting time so that their weekly profit ‘one week after the fad diet took hold’ was 8% less than $200,000.)

10. A (large) company made a daily profit of $10,000. Because of some positive media attention, they expect their daily profit to increase by 1% a day for 2 weeks.

   (a) How much profit did the company make over these 14 days?
   (b) How much extra profit did the company make as a result of the media attention? (Assume that without the media attention, they would have kept making $10,000 a day.)