Arithmetic Sequences

• Recall an Arithmetic Sequence is a sequence where the difference between any two consecutive numbers in the sequence is constant.

\[ a_{k+1} - a_k = d \] where \( d \) is a constant.

• Examples:

1. 1, 4, 7, 10, 13, … IS arithmetic, with constant difference \( d = 3 \)
2. 2, 4, 8, 16, 32, … is NOT arithmetic
3. −3, 7, 17, 27, … IS arithmetic, with constant difference \( d = 10 \)

• Real World Examples:

1. You start a new job and you’re told you salary is $29,000 for the first year, and that you’ll get a $1700 raise each year. What will your salary be in the third year? What will your salary be in 10 years? How long does it take for your salary to (at least) double?

2. A new company has a loss of $2,500 in its first month, but they expect their monthly profit to increase by $400 each month. What is their profit in the 12th month? What is their total profit/loss of the year?

Both these scenarios can be modeled by Arithmetic Sequences, and we will develop tools to help us answer these questions.
General Way to Write an Arithmetic Sequence

• Consider the Arithmetic Sequence below. Notice the first term is 5 and the common difference is 2:

\[ 5, 7, 9, 11, 13, \ldots \]

We’ll look at the pattern that the common difference of 2 creates.

\[ \begin{array}{cccc}
 5, & 7, & 9, & 11, \\
a_1, & a_2, & a_3, & a_4, \\
5, & 5 + (1)2, & 5 + (2)2, & 5 + (3)2, \\
a_1, & a_2, & a_3, & a_4, \\
13, & \ldots \\
a_5 \\
\end{array} \]

We notice the pattern for this sequence \( a_n = 5 + (n - 1)2 \)

We also see that \( a_n = a_{n-1} + 2 \) (each term is 2 more than the previous term)

• **Way to Write a Formula for an Arithmetic Sequence:**
  Given that \( a_1, a_2, a_3, \ldots \) is an arithmetic sequence with common difference \( d \),
  We can rewrite the sequence as
  \[ a_n = a_1 + (n - 1)d \]
  where the index starts at \( n = 1 \). Here \( a_1 \) is the first term of the sequence (a constant) and \( d \) is the common difference (also a constant).
Examples (Arithmetic Sequences)

1. Given the Arithmetic Sequence $-10, -4, 2, 8, \ldots$

   *To understand everything about this sequences we need to know:*

   - It’s Arithmetic
   - With common difference $d = 6$
   - And first term $a_1 = -10$

(a) Find the fifth term in the sequence.

   *Let’s agree to start this sequence at $n = 1$ so we can call the first term $a_1$*

   *Since the first 4 terms are given, and the common difference is $d = 6$, we can see the $5^{th}$ term 6 more than $4^{th}$ term.*

   *i.e. $a_5 = a_4 + 6 = 8 + 6 = 14$*

(b) Find the $20^{th}$ term in the sequence.

   *Use the formula: $a_n = a_1 + (n - 1)d$*

   

   $a_n = -10 + (n - 1)6$ with starting term $n = 1$

   *This mean the $20^{th}$ term is*

   

   $a_{20} = -10 + (20 - 1)6 = -10 + 19 \cdot 6 = -10 + 114 = 104$

(c) Find a formula for the $n^{th}$ term in the sequence.

   *We did that above because we liked the shortcut.*
Examples, Real World Arithmetic Sequences (Number 1)

Joan invests $3,000 in an account that pays 2% simple interest. Determine how much money is in her account after each of the first 5 years.

- Using $I = PRT$ formula for simple interest.
  
  \[
  P = $3,000 \\
  R = 0.02 \\
  T = (\text{depends which year we’re talking about})
  \]

<table>
<thead>
<tr>
<th>Year</th>
<th>Interest (I = PRT)</th>
<th>Total In Account</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$3000 \cdot 0.02 \cdot 1 = $60$</td>
<td>$3000 + $60 = $3060$</td>
</tr>
<tr>
<td>2</td>
<td>$3000 \cdot 0.02 \cdot 2 = $120$</td>
<td>$3000 + $120 = $3120$</td>
</tr>
<tr>
<td>3</td>
<td>$3000 \cdot 0.02 \cdot 3 = $180$</td>
<td>$3000 + $180 = $3180$</td>
</tr>
<tr>
<td>4</td>
<td>$3000 \cdot 0.02 \cdot 4 = $240$</td>
<td>$3000 + $240 = $3240$</td>
</tr>
<tr>
<td>5</td>
<td>$3000 \cdot 0.02 \cdot 5 = $300$</td>
<td>$3000 + $300 = $3300$</td>
</tr>
</tbody>
</table>

- Another way to think about it:
  The simple interest from each year is $3000 \cdot 0.02 \cdot 1 = \$60$, so each year Joan has $\$60$ more than the previous year.

- This looks like an arithmetic sequence.
  With starting value $a_1 = \$3060$ and common difference $d = \$60$.

  $3060, 3120, 3180, 3240, 3300, \ldots$

- So the amount of money in the account at (the end of) year $n$ is

  \[
  a_n = 3060 + (n - 1)\times60
  \]
Examples, Real World Arithmetic Sequences (Number 2)
You start a new job and you’re told your salary is $29,000 for the first year, and that you’ll get a $1700 raise each year. What will your salary be in the third year? What will your salary be in 10 years? How long does it take for your salary to (at least) double?

1. Fill in the table indicating your salary in the first several years:

<table>
<thead>
<tr>
<th>Year</th>
<th>Salary in indicated year</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

2. Notice that the list of your salaries year by year look like an Arithmetic Sequence. Identify the common difference, and the first term:

\[ a_1 = \]
\[ d = \]

This is the important bit. You make the table to help you with this.

3. Write a formula for \( a_n \), your salary in year \( n \).

\[ a_n = \]

Where \( n \) is measured in years, and \( a_n \) is your salary in year \( n \) (measured in dollars)

4. What will your salary be in the third year?

\[ a_3 = \$29,000 + (3 - 1)\$1700 = \$32,400 \]

5. What will your salary be in 10 years?

\[ a_{10} = \$29,000 + (10 - 1)\$1700 = \$44,300 \]
6. How long does it take for your salary to (at least) double?

Double your (starting) salary is $58,000

\[
\begin{align*}
58,000 &= 29,000 + (n - 1)1700 \\
58,000 - 29,000 &= (n - 1)1700 \\
29,000 &= n \cdot 1700 - 1700 \\
29,000 + 1700 &= n \cdot 1700 \\
30,700 &= n \cdot 1700 \\
\frac{30,700}{1700} &= n
\end{align*}
\]

So \( n \approx 18.0588 \)

So a little more than 18 years. We can check:

\[
a_{18} = 29,000 + (18 - 1)1700 = 57,900 \text{ (less than double)}
\]

\[
a_{19} = 29,000 + (19 - 1)1700 = 59,600 \text{ (more than double)}
\]
Partial Sums of an Arithmetic Sequence

- Remember $S_n$ is the sum of the first $n$ terms of a sequence.

- **Working out the $n^{th}$ partial sum of an Arithmetic Sequence.**
  Here’s one way to write our Arithmetic sequence:

  \[ a_1, (a_1 + d), (a_1 + 2d), (a_1 + 3d), \ldots \]

  - So the $n^{th}$ Partial Sum of the Arithmetic Series can be written as:

    \[
    S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (n-3)d) + (a_1 + (n-2)d) + (a_1 + (n-1)d)
    \]

- Another way to name the terms:

  \[
  S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (n-3)d) + (a_1 + (n-2)d) + (a_1 + (n-1)d)
  \]

  - This gives us another way to write $S_n$

    \[
    S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_n - (n-3)d) + (a_n - (n-2)d) + (a_n - (n-1)d)
    \]

  - Add the two lines together

    \[
    S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (n-2)d) + (a_1 + (n-1)d)
    \]

    \[
    S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_n - (n-2)d) + (a_n - (n-1)d)
    \]

    \[
    2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) + (a_1 + a_n)
    \]

    We count the $(a_1 + a_n)$ terms on the right to see that $2S_n = n(a_1 + a_n)$ and

    \[
    S_n = \frac{n(a_1 + a_n)}{2} = \frac{n}{2}(a_1 + a_n)
    \]
Using the Formula the $n^{th}$ Partial Sum of an Arithmetic Sequence

- The $n^{th}$ partial sum of an Arithmetic Sequence $a_1, a_2, a_3, \ldots$ is given by

\[ S_n = \frac{n}{2} (a_1 + a_n) \]

Where $a_1$ is the first term of the Arithmetic Sequence and $a_n$ is the $n^{th}$ term of the Arithmetic Series.

- **Questions:** The Arithmetic Sequence 7, 10, 13, 16, 19, 22, \ldots

1. Find the 4$^{th}$ Partial Sum of the Sequence

\[ S_4 = \frac{4}{2} (a_1 + a_4) = \frac{4}{2} (7 + 16) = 2(23) = 44 \]

(We can double check that $7 + 10 + 13 + 16 = 46$)

2. Find the 20$^{th}$ term of the Sequence

*Our Arithmetic Sequence has $a_1 = 7$ and $d = 3$ so $a_n = 7 + (n-1)3$*

\[ a_{20} = 7 + (20 - 1)3 = 7 + 19 \cdot 3 = 64 \]

3. Find the 20$^{th}$ Partial Sum of the Sequence

\[ S_{20} = \frac{20}{2} (7 + 64) = 10(71) = 710 \]

*Which is much faster than*

\[ 7 + 10 + 13 + 16 + 19 + 22 + 25 + 28 + 31 + 34 + 37 + 40 + 43 + 46 + 49 + 52 + 55 + 58 + 61 + 64 = 710 \]
Examples, Real World Arithmetic Sequences (Number 3)

A new company has a loss of $2,500 in its first month, but they expect their monthly profit to increase by $400 each month. What is their profit in the 12th month? What is their total profit/loss of the year?

<table>
<thead>
<tr>
<th>Month</th>
<th>Profit/Lost for Month in indicated month</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-2,500</td>
</tr>
<tr>
<td>2</td>
<td>$-2,100</td>
</tr>
<tr>
<td>3</td>
<td>$-1,700</td>
</tr>
</tbody>
</table>

This is an Arithmetic Sequence with $a_1 = -2500$ and $d = 400$

So $a_n$ represents the monthly profit/loss in month $n$ and

$$a_n = -$2500 + (n - 1)400$$

- The profit in the 12th month is represented by $a_{12}$

$$a_{12} = -$2500 + (12 - 1)400 = 1900$$

- The total profit/loss for the year is the

$$(\text{profit/loss for Jan}) + (\text{profit/loss for Feb}) + \cdots + (\text{profit/loss for Dec})$$

which can be represented in symbols as

$$a_1 + a_2 + \cdots + a_{12} = S_{12}$$

So Total Profits for the Year

$$S_{12} = \frac{12}{2}(a_1 + a_{12}) = \frac{12}{2}(-2500 + 1900) = 6(-600) = -$3600$$

They lost a total of $3600 for the year.

Much faster than $-2500 - 2100 - 1700 - 1300 - 900 - 500 - 100 + 300 + 700 + 1100 + 1500 + 1900 = -3600$
Examples, Real World Arithmetic Sequences (Number 2...again)
You start a new job and you’re told you salary is $29,000 for the first year, and that you’ll get a $1700 raise each year. How much money will you make total your first 10 years on the job.

- We saw earlier this is an Arithmetic sequence with $a_1 = 29,000$, $d = 1700$
  
  $$a_n = 29000 + (n - 1)1700$$
  
  $$a_{10} = 29000 + (10 - 1)1700 = 44,300$$

- Total you make in the first 10 years is $S_{10}$
  
  $$S_{10} = \frac{n}{2}(a_1 + a_{10})$$
  
  $$S_{10} = \frac{10}{2}(29000 + 44300) = 366,500$$
Homework (Arithmetic Sequences)

1. For each of the Arithmetic sequences below:
   - Find the 5th term of the sequence.
   - Find the 25th term of the sequence.
   - Find a formula for \( a_n \), the \( n^{th} \) term of the sequence.

(a) 3, 5, 7, 9, \ldots
(b) 3, 8, 13, 18, \ldots
(c) −7, −6, −5, −4, \ldots
(d) −1, 6, 13, 20, \ldots

2. Find the 6th Partial Sum of the Arithmetic Sequence 3, 5, 7, 9, \ldots

3. Find the 15th Partial Sum of the Arithmetic Sequence −7, −6, −5, −4, \ldots

4. Find the 9th Partial Sum of the Arithmetic Sequence 3, 8, 13, 18, \ldots

5. Find the 100th Partial Sum of the Arithmetic Sequence −1, 6, 13, 20, \ldots

6. Kevin loves to read. He currently has 150 novels in his house. Every month he buys 4 new novels.
   
   (a) Fill in the table below:
   
<table>
<thead>
<tr>
<th>Month</th>
<th>Number of Novels Kevin has in indicated month</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>150</td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

   (b) Write a formula for \( a_n \), the number of novels Kevin has in his house on month \( n \).
   (c) How many novels does Kevin have in his house on month 10?
   (d) How many novels does Kevin have in his house after 5 years (60 months).
7. You make an investment of $500 that pays $\frac{1}{4}\%$ simple interest each month.
   
   (a) How much in interest do you make in month 1?
   (b) How much is the investment worth in month 1?
   (c) What is the investment worth in month 12?
   (d) Write a formula for $a_n$, the value of the investment in month $n$.

8. Your parents are going to sell you their old car. Since you just graduated, they know you’re a little short on cash at the moment. They agree your payment this month will be $30, and that each month your monthly payment will increase by $5.

   Call this month ‘month 1’

   (a) What is your monthly payment next month (month 2)?
   (b) Write a formula for $a_n$, your monthly payment in month $n$.
   (c) What is your monthly payment on month 12?
   (d) What is your monthly payment on month 36?
   (e) How much in total have you paid your parents after a year?
   (f) How much in total have you paid your parents after 3 years?

9. This week (week 1) your small business made a profit of $-200$ (also known as a loss). You anticipate each week that your weekly profit will increase by $9$.

   (a) What is your weekly profit in week 2?
   (b) What is your weekly profit at the end of the year (in week 52)?
   (c) How much money did you make (or lose) total this year?
   (d) Which week did you break even (make a weekly profit of $0$)?

10. Sadly, Tony is addicted to drugs. This week (week 1) he spent $10 on his habit. We all know drugs are addicting and each week addicts need a little more of the drug to get the same effect. So we know that each week Tony will spend $1$ more on drugs than he spent the previous week.

    (a) How much does Tony spend on drugs during week 4?
    (b) How much does Tony spend on drugs during week 52?
    (c) At the end of the year (52 weeks) how much in total has Tony spent on drugs?