• **Inverse Functions**

Suppose $f$ and $g$ are two functions such that

1. $(g \circ f)(x) = x$ for all $x$ in the domain of $f$ and
2. $(f \circ g)(x) = x$ for all $x$ in the domain of $g$

then $f$ and $g$ are **inverses** of each other and the functions $f$ and $g$ are said to be **invertible**.

• What do we think should be the inverse of the function $f$ that takes an input divides by 2 then adds 3?

Check your guess.

• Not all functions have inverses.

• **Properties of inverse functions:** Suppose $f$ and $g$ are inverse functions.

  – The range of $f$ is the domain of $g$ and the domain of $f$ is the range of $g$
  – $f(a) = b$ if and only if $g(b) = a$
  – The point $(a, b)$ is on the graph of $f$ is and only if $(b, a)$ is on the graph of $g$.

• **Uniqueness of Inverse Functions and Their Graphs:** Suppose $f$ is an invertible function.

  – There is exactly one inverse function for $f$, denoted $f^{-1}$ (read $f$-inverse)
  – The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across the line $y = x$. 
Examples

1. Given $f$ is invertible and $f(-3) = 1$, $f(2) = -3$, and $f(4) = 2$
   (a) What is $f^{-1}(-3)$?
   (b) Given that $f(x) = 4x - 1$ is invertible, what is $f^{-1}(7)$?

A function $f$ is said to be one-to-one if $f$ matches different inputs to different outputs. Equivalently, $f$ is one-to-one if and only if whenever $f(c) = f(d)$, then $c = d$.

How to show a function is one-to-one

- Analytically
  * Assume $f(c) = f(d)$
  * using only operations which are reversible simplify the above equation
    (Make sure you don’t introduce extraneous solutions or lose solutions, etc.)
  * If you can conclude $c = d$, then the function is one-to-one.

- Graphically
  The Horizontal Line Test: A function $f$ is one-to-one if and only if no horizontal line intersects the graph of $f$ more than once.

Showing a function is NOT one-to-one

- Analytically
  Guess and check 2 different inputs that produce the same output.
- Graphically
  Show the graph fails the horizontal line test.

Equivalent Conditions for Invertibility: Suppose $f$ is a function. The following statements are equivalent.

- $f$ is invertible
- $f$ is one-to-one
- The graph of $f$ passes the Horizontal Line Test
• Examples Determine analytically if the following functions are one-to-one or not.

1. \( y = x^2 + 3 \)
2. \( y = 2(x - 3) \)

• Steps for finding the Inverse of a One-to-one Function

- Write \( y = f(x) \)
- Interchange \( x \) and \( y \)
- Solve \( x = f(y) \) for \( y \) to obtain \( y = f^{-1}(x) \)

• Examples: Find the inverse of the following one-to-one functions. Check your answers analytically using function composition

1. \( f(x) = \frac{1 - 2x}{3} \)
2. \( f(x) = \frac{-x}{5 + 4x} \)

• Examples Graph the following functions to show they are one-to-one and find their inverses. Check your answers analytically using function composition and graphically.

1. \( y = x^2 + 2x - 3, \quad x \geq -1 \)
2. \( y = \sqrt{x + 2} \)