A Rational Function is a function which is the ratio of polynomial functions. Said differently, $r$ is a rational function if it is of the form

$$r(x) = \frac{p(x)}{q(x)}$$

where $p$ and $q$ are polynomial functions.

Example: Find the domain of the following rational functions, and write them in the form $\frac{p(x)}{q(x)}$ and simplify.

1. $f(x) = \frac{(x - 2)(x + 3)}{(x^2 - 4)(x + 1)}$
2. $f(x) = \left(\frac{3 + x}{x^2 - 1}\right) \div \frac{4x + 1}{x^2 - 1}$
• Asymptotes

- The line \( x = c \) is called a **vertical asymptote** of the graph of a function \( y = f(x) \) if as \( x \to c^- \) or as \( x \to c^+ \), either \( f(x) \to \infty \) or \( f(x) \to -\infty \).

- The line \( y = c \) is called a **horizontal asymptote** of the graph of a function \( y = f(x) \) if as \( x \to -\infty \) or as \( x \to \infty \), \( f(x) \to c \).

• **Examples:** The graph of \( f(x) \) is given below. Determine if there are any vertical or horizontal asymptotes.

3. \( f(x) = \frac{2(x + 1)^2}{(x - 2)(x + 4)} \)

4. \( f(x) = \frac{3x + 9}{x^2 - 9} \)
• **Determining Vertical Asymptotes (and Holes) from the equation of a rational function**

Suppose \( r \) is a rational function which can be written as \( r(x) = \frac{p(x)}{q(x)} \) where \( p \) and \( q \) have no common zeros.

(i.e. \( r(x) = \frac{p(x)}{q(x)} \) is in lowest terms)

Let \( c \) be a real number which is *not* in the domain of \( r \).

- If \( q(c) \neq 0 \), then the graph of \( y = r(x) \) has a hole at \((c, \frac{p(c)}{q(c)})\), aka \((c, r(c))\)
- If \( q(c) = 0 \), then the line \( x = c \) is a vertical asymptote of the graph of \( y = r(x) \).

• **Example:** Find the vertical asymptotes of, and/or holes in, the graphs of the following rational functions.

5. \( r(x) = \frac{x^2 + x - 6}{(2x^2 + 8x - 24)(3x - 7)} \)

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• **Determining Horizontal Asymptotes from the equation of a rational function:**

Suppose \( r \) is a rational function and \( r(x) = \frac{p(x)}{q(x)} \), where \( p \) and \( q \) are polynomial functions with leading coefficients \( a \) and \( b \), respectively.

- If the degree of \( p(x) \) is the same as the degree of \( q(x) \), then \( y = \frac{a}{b} \) is the horizontal asymptote of the graph of \( y = r(x) \).
- If the degree of \( p(x) \) is less than the degree of \( q(x) \), then \( y = 0 \) is the horizontal asymptote of the graph of \( y = r(x) \).
- If the degree of \( p(x) \) is greater than the degree of \( q(x) \), then the graph of \( y = r(x) \) has no horizontal asymptotes.

• **Example:** List the horizontal asymptotes, if any, of the graphs of the following functions.

6. \( f(x) = \frac{(2x^2 - 1)(3x + 1)}{7x - 5x^2} \)
7. \( r(x) = \frac{3x^3 + 2x - 1}{(2x - 1)(x + 3)(5x - 4)} \)
8. \( g(x) = \frac{x - 1}{x^2 - 1} \)
Example:

- List the domain of the function
- Find all vertical asymptotes
- Find all holes in the graph
- Find the horizontal asymptote (if it exists)

9. \( f(x) = \frac{4x^2 + 12x}{x^2 + 6x + 9} \)

10. (optional) \( f(x) = \frac{7x^3 + 3x^2 - x + 12}{x^2 - 6x + 13} \)

11. (optional) \( f(x) = \frac{2x^2 - 4x - 16}{x^3 - 4x^2 + 2x - 8} \)