• Powers of Diagonalizable Matrices

• Example:

1. For a diagonal matrix \( D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \), find \( D^k \).

2. Let \( A \) be an \( n \times n \) diagonalizable matrix. So there exists an invertible matrix \( P \) and diagonal matrix \( D \) such that \( P^{-1}AP = D \), find a simple expression for \( A^k \), for \( k \) a positive integer.

• Example:

3. For \( D = \begin{bmatrix} -5 & -3 & -3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix} \), find \( D^{15} \). Note that we have seen this matrix before.

4. For \( A = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} \), find \( A^{20} \).
• Differential Equations

• Let $x_1(t)$, $x_2(t)$, ..., $x_n(t)$ be $n$ functions of $t$ (time), and let $a_{ij}$ for $1 \leq i, j \leq n$ be scalars. Then

\[
\begin{align*}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\
&\vdots \quad \vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n
\end{align*}
\]

Is a system of homogeneous, ordinary, linear differential equations with constant coefficients.

For $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$

We use the notation $x' = Ax$, where the derivative is understood to be with respect to $t$. 
• Solutions to a system of homogeneous, ordinary, linear differential equations with constant coefficients.

- For $x' = Ax$
- Consider the trial solution $x = ve^{\lambda t}$ as we did in section 9.1.
- We’ll need to determine the vector $v$ and the constant $\lambda$.
- Substituting the trial solution in to the system (as we did in Section 9.1) we see we need

$$\lambda ve^{\lambda t} = A(ve^{\lambda t}) = (Av)e^{\lambda t}$$

- So $v$ and $\lambda$ must satisfy:

$$Av = \lambda v$$

- You need to find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct) with corresponding eigenvectors $v_1, v_2, \ldots, v_n$, then...

- The General Solution to the differential equation is:

$$x = c_1v_1e^{\lambda_1t} + c_2v_2e^{\lambda_2t} + \cdots + c_nv_ne^{\lambda_nt}$$

- If we’re given an initial condition

$$x(0) = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

or $x_1(0) = y_1, x_2(0) = y_2, \ldots, x_n(0) = y_n$

we will use this information to find all the constants $c_1, \ldots, c_n$. This is called the particular solution.
• Example:

5. For the system of differential equations:

\[
\begin{align*}
\frac{dx}{dt} &= x - 4y \\
\frac{dy}{dt} &= 2x - 7y
\end{align*}
\]

(a) Find the general solution.

(b) Find the particular soln. subject to the initial condition \( x(0) = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \)

6. For \( A = \begin{bmatrix}
-3 & -1 & -1 \\
0 & 2 & 0 \\
5 & 1 & 3
\end{bmatrix} \)

(a) Find the general solution to the system of differential equations:

\[ x' = Ax. \]

(b) Find the particular solution to this system subject to the initial condition \( x(0) = \begin{bmatrix} -5 \\ 10 \\ -1 \end{bmatrix} \)

7. (optional)

\[
\begin{align*}
\frac{dx}{dt} &= -7x + 4z \\
\frac{dy}{dt} &= -8x + 2y + 3z \\
\frac{dz}{dt} &= -8x + 5z
\end{align*}
\]

(a) Find the general solution to the system of differential equations.

(b) Find the particular solution to this system subject to the initial condition \( x(0) = 2, y(0) = 0, z(0) = 5 \)