9.2 Diagonal Form of a Matrix
Part 2

• Recall:
  – Last time we showed
    * If $A$ has $n$ linearly independent eigenvectors, then $A$ is diagonalizable
    * If $A$ is diagonalizable then $A$ has $n$ linearly independent eigenvectors.
  – So $A$ is diagonalizable if and only if it has $n$ linearly independent eigenvectors.
  – And we saw not all $n \times n$ matrices are diagonalizable.

• Complex Eigenvalues
  – The Characteristic polynomial of an $n \times n$ real matrix is a polynomial of degree $n$ (with real coefficients).
  – The Fundamental Theorem of Algebra tells us that if $p(\lambda)$ is a polynomial of degree $n$ with real coefficients, then it has (exactly) $n$ complex roots counted with multiplicities.
  – This means that if we allow our scalars to be complex numbers, every $n \times n$ matrix has $n$ eigenvalues counted with multiplicities.

• Example

  1. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
     (a) Find the characteristic equation of $A$.
     (b) Find the complex eigenvalues of $A$.
     (c) Find the (complex) eigenvectors of $A$. 
The Trace and the Characteristic Polynomial

Recall: For an \( n \times n \) matrix \( A \), the \textbf{trace} of \( A \) is the sums of its diagonal elements, and is denoted \( \text{tr}(A) \).

Properties of the Trace:
Let \( A, B \) be \( n \times n \) matrices, we have

\[ \begin{align*}
(1) \quad \text{tr}(A + B) &= \text{tr}(A) + \text{tr}(B) \\
(2) \quad \text{For any scalar } c, \quad \text{tr}(cA) &= c \text{tr}(A) \\
(3) \quad \text{tr}(AB) &= \text{tr}(BA)
\end{align*} \]

Examples:

2. Find a counterexample to \( \text{tr}(AB) = \text{tr}(A) \text{tr}(B) \) for \( A, B \ 2 \times 2 \) matrices.

Examples:

3. For \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), calculate the characteristic polynomial of \( A \).
4. Can you express your answer to (3) in terms of familiar quantities?

Theorem: Let \( A \) be an \( n \times n \) matrix, and let \( p(\lambda) = \det(A - \lambda I) \), then

\[ p(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} \text{tr}(A) \lambda^{n-1} + \cdots + \det(A). \]

Examples

5. Prove the above theorem.
• Invertible Matrices and Eigenvalues/Eigenvectors

• **Theorem:** A square matrix \( A \) is invertible if and only if all of its eigenvalues are non-zero.

• **Theorem:** Let \( A \) be an \( n \times n \) matrix with eigenvalues

\[
\lambda_1, \lambda_2, \ldots, \lambda_n
\]

whose corresponding eigenvectors are

\[
x_1, x_2, \ldots, x_n
\]

respectively. Then for any positive integer \( k \) the matrix \( A^k \) has eigenvalues

\[
\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k
\]

with the same corresponding eigenvectors \((x_1, x_2, \ldots, x_n)\).

If \( A \) is invertible, the above is also true for negative integers: The eigenvalues of \( A^{-1} \) are

\[
\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}
\]

with the same corresponding eigenvectors \((x_1, x_2, \ldots, x_n)\).

• **Examples:**

6. Prove the above theorem.