7.2 The Matrix of a Linear Transformation

• Ordered Basis
Let $V$ be an $n$ dimensional vectors space.
Let $\mathcal{B} = \{v_1, v_2, \ldots, v_k\}$ be a basis for $V$. We fix the order of the elements $v_i$ in the basis $\mathcal{B}$, and call it an ordered basis for $V$.

Coordinate Vector
For $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ a ordered basis of the $n$ dimensional vector space $V$, any vector $v$ in $V$ can be uniquely written as a linear combination of $v_1, v_2, \ldots, v_n$:

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$ 

The coordinate vector of $v$ relative to the ordered basis $\mathcal{B}$ is

$$[v]_\mathcal{B} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$ 

• Examples
1. Let $\mathcal{B} = \{1, x, \ldots, x^4\}$ be the standard ordered basis for $\mathcal{P}_4$.
   (a) Let $p(x) = 2 - x + 3x^3 - 11x^4$, write $[p(x)]_\mathcal{B}$
   (b) Let $q(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$, write $[q(x)]_\mathcal{B}$

2. Let $\mathcal{B} = \{e_1, e_2, e_3\}$ be the standard ordered basis for $\mathbb{R}^3$, and let $x =$
   $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ 
   write $[x]_\mathcal{B}$

3. $\mathcal{B}' = \left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ be an ordered basis for $\mathbb{R}$,
   (a) Let $v = \begin{bmatrix} 10 \\ 9 \\ 8 \end{bmatrix}$, write $[v]_{\mathcal{B}'}$
   (b) Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, write $[x]_{\mathcal{B}'}$
• Matrix of a Linear Transformation
Let $V$ and $W$ be finite dimensional vector spaces and let $T : V \to W$ be a linear transformation. Let $\dim(V) = n$ and $\dim(W) = m$. Furthermore let

$$B = \{v_1, v_2, \ldots, v_n\} \text{ and } B' = \{w_1, w_2, \ldots, w_m\}$$

be ordered bases for $V, W$ respectively. The **matrix representation of $T$ relative to the ordered basis $B$ and $B'$**, denoted by $[T]_{B'}^B$, is defined to be the matrix whose $j$-th column is the coordinate vector $[T(v_j)]_{B'}$.

• The Big Ideas

- So, given ANY linear transformation $T : V \to W$, you can write a matrix so that $T$ is represented by $T_A$
  
  * You just need to pick ordered bases so that the elements of $V, W$ can be written as column vectors.
  * All this mean the transformation $T(v)$ and the matrix multiplication $T_A[v]_{B'}$ give the same answer (except the second one is coordinate vector, the first may be in a different format).

- To know everything about a linear transformation, you only need to know what it does to a basis.
  
  * since ever vector in a vector space can be uniquely written as a linear combination of basis vectors.
  * and since linear transformations are ‘nice to’ linear combinations.

• Finding the matrix representation of a linear transformation

- Take each vector $v_j$ in the basis for $V$
- Do the linear transformation $T$ to $v_j$ (find $T(v_j)$
- Write the result as a coordinate vector with respect to your ordered basis for $W$
- Write the resulting column vectors in the correct order, this matrix is $[T]$
4. Let $T: \mathbb{R}^2 \to \mathbb{R}^4$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ 0 \\ 2x - 3y \\ x + y \end{bmatrix}$$

and let $B^2$ be the standard ordered basis for $\mathbb{R}^2$ and $B^4$ be the standard ordered basis for $\mathbb{R}^4$.

(a) Find $[T] = [T]_{B^4}^{B^2}$

(b) For $v = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, find $[v]_{B^2}$

(c) Do the matrix multiplication $[T][v]_{B^2}$

(d) Compute $T(x)$ using the original definition of $T$

(e) Do your answers to (4c) and (4d) agree?

5. Let $T: M_{2 \times 2}(\mathbb{R}) \to \mathcal{P}_2$ be the linear transformation given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + (b - c)x + ax^2$$

And let $B$ be the standard basis for $M_{2 \times 2}(\mathbb{R})$ and $B'$ be the standard basis for $\mathcal{P}_2$

(a) Find $[T] = [T]_{B'}^{B'}$

(b) For $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find $[C]_{B}$

(c) Do the matrix multiplication $[T][C]_{B^2}$

(d) Compute $T(C)$ using the original definition of $T$

(e) Do your answers to (5c) and (5d) agree?

6. $T: M_{2 \times 2}(\mathbb{R}) \to \mathbb{R}^3$ is a linear transformation and has the property that

$$T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

Given that $\left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$ is a basis for $M_{2 \times 2}(\mathbb{R})$, find $T\left(\begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}\right)$. 
Nice Transformations in $\mathbb{R}^2$

• **Rotations by $\theta$**
  Let $R_{\theta} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation that rotates every vector counterclockwise (around the origin) by the angle $\theta$.

• **Example**
  7. Write $T_{R_{\theta}} = [R_{\theta}]$ with respect to the standard basis for $\mathbb{R}^2$.

• **Reflection across the line $y = x$**
  Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation that reflects points in $\mathbb{R}^2$ across the line $y = x$.

• **Example**
  8. Explicitly write the transformation $T$.
  9. What is the matrix of $T$ relative to the standard basis for $\mathbb{R}^2$?

• **Projections**
  $P : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is projection onto the $x$-axis if $P \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x \\ 0 \end{bmatrix}$

• **Example**
  10. What is the matrix of $P$ relative to the standard basis for $\mathbb{R}^2$?
‘Putting Transformations Together’

- **Sum and Scalar Multiple of a Linear Transformation**
  Let $V$ and $W$ be two finite dimensional vector spaces with basis $\mathcal{B}$ and $\mathcal{B}'$ respectively. Let $S : V \rightarrow W$ and $T : V \rightarrow W$ be two linear transformations whose matrices with respect to the given bases are $[S]$ and $[T]$ respectively. Let $v$ be any vector in $V$:
  - The **sum** $S + T$ of the two linear transformation is defined by
    $$(S + T)(v) = S(v) + T(v)$$
  - The **scalar multiple** of the linear transformation $T$ is defined by
    $$(cS)(v) = cS(v)$$
  Then $S + T : V \rightarrow W$ and $cS : V \rightarrow W$ are both linear transformations. Furthermore the matrices of the linear transformations $S + T$ and $cS$ with respect to the given bases are given by $[S + T] = [S] + [T]$ and $[cS] = c[S]$.

- **Composition of Linear Transformations**
  Let $U, V, W$ be finite dimensional vector spaces with ordered bases $B, B', B''$ respectively, and let $T : U \rightarrow V$ and $S : V \rightarrow W$ be linear transformations. We define the composition of the two linear transformations $S \circ T : U \rightarrow W$ by
  $$S(\circ T)(u) = S(T(u))$$
  where $u$ is any vector in $U$. Then $S \circ T : U \rightarrow W$ is also a linear transformation and the matrix of the linear transformation $S \circ T$ is given by the product of the matrices of $S$ and $T$ (relative to the appropriate basis). Precisely,
  $$[S \circ T]_{\mathcal{B}''}^{\mathcal{B}'} = [S]_{\mathcal{B}''}^{\mathcal{B}'} [T]_{\mathcal{B}}^{\mathcal{B}'}$$

- **Example**
  11. Consider the transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that reflects vectors across the $x$-axis then rotates them by $\frac{\pi}{3}$ radians ($60^\circ$)
    (a) Find a matrix that represents reflecting vectors in $\mathbb{R}^2$ over the $x$-axis with respect to the standard basis.
    (b) Find a matrix that represents rotating vectors in $\mathbb{R}^2$ by $\frac{\pi}{3}$ radians (with respect to the standard basis)
    (c) Find the matrix that represents $T$ (with respect to the stand basis) directly (by finding $T(e_1)$ and $T(e_2)$
    (d) Find the matrix that represents $T$ (with respect to the stand basis) by multiplying the matrices from parts $a, b$
    (e) Do the two answers agree?