7.1 Linear Transformations Associated to Matrices

- Map/Function from one vector space $V$ to another vector space $W$

$$T : V \rightarrow W$$

- Example $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by $T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x \\ x + y \\ y \end{bmatrix}$

- Example $T : \mathcal{P}_3 \rightarrow \mathbb{R}^4$ given by $T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a + b \\ 0 \\ d \\ 1 \end{bmatrix}$

- Example $T : \mathcal{M}_{2 \times 2}(\mathbb{R}) \rightarrow \mathcal{P}_2$ given by $T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a + (d^2)x + (bc)x^2$

- **Linear Transformation**

  Let $V$ and $W$ be real (or complex) vector spaces. A function $T : V \rightarrow W$ is called a linear transformation if for all vectors $\mathbf{u}, \mathbf{v}$ in $V$ and for all scalars $c$, $T$ satisfies the following two conditions

  - (1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
  - (2) $T(c \mathbf{u}) = cT(\mathbf{u})$.

- If $A$ is an $m \times n$ real (or complex) matrix, then $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is defined by $T(\mathbf{x}) = A\mathbf{x}$ is always a linear transformation.
• Showing a Transformation $T$ IS a Linear Transformation
  For $T : V \rightarrow W$
   - Name 2 arbitrary objects is $V$ ($u$ and $v$) and one arbitrary scalar $c$.

   - Show Condition 1: $T(u + v) = T(u) + T(v)$
     * Start with the LHS, and simplify
     * Start with the RHS and simplify
     * Determine/conclude they give you the same object in $W$.

   - Show Condition 2: $T(cu) = cT(u)$
     * Start with the LHS, and simplify
     * Start with the RHS and simplify
     * Determine/conclude that they give you the same object in $W$.

   - If you satisfy both conditions, conclude $T$ is a linear transformation.

• Showing a Transformation $T$ IS NOT a Linear Transformation
  For $T : V \rightarrow W$
   - Do either of the following:
     - Show Condition 1 is violated $T(u + v) = T(u) + T(v)$
       * Find (guess/check) two vectors $u, v$ in $V$:
       * Start with the RHS and simplify
       * Start with the RHS and simplify
       * Determine/show they do NOT give you the same object in $W$

     - Show Condition 2 is violated: $T(cu) = cT(u)$
       * Find (guess/check) one vector $u$ in $V$ and once scalar 0:
       * Start with the RHS and simplify
       * Start with the RHS and simplify
       * Determine/show they do NOT give you the same object in $W$

   - Once you do either of these, conduce that $T$ is NOT a linear transformation.
• **Examples** For each of the following functions, determine if the function is a linear transformation or not.

1. \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) given by \( T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x \\ x + y \\ y \end{bmatrix} \)

2. \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) given by \( T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \)
• **Thm:**

Let $V$ and $W$ be real (or complex) vector spaces. The following are equivalent for a function $T : V \rightarrow W$.

- (1) $T$ is a linear transformation.
- (2) For all vectors $u, v$ in $V$ and for any scalar $c$, $T(cu + v) = cT(u) + T(v)$.
- (3) For all vectors $v_1, v_2, \ldots, v_k$ in $V$ and for all scalars $c_1, c_2, \ldots, c_k$ we have
  \[ T(c_1v_1 + c_2v_2 + \ldots + c_kv_k) = c_1T(v_1) + c_2T(v_2) + \cdots + c_kT(v_k). \]

• **Theorem**

Let $V$ and $W$ be vector spaces and let $T : V \rightarrow W$ be a linear transformation. If $0_V$ is the zero vector of $V$ and if $0_W$ is the zero vector of $W$, then

\[ T(0_V) = 0_W \]

• **Range of A Linear Transformation**

Let $V$ and $W$ be vector spaces and let $T : V \rightarrow W$ be a linear transformation. The range of $T$, denoted by range($T$), consists of all vectors $w$ in $W$, such that $w = T(v)$, for some vector $v$ in $V$.

- The range($T$) is always a subspace of $W$ (can you prove it).
- If $T_A$ is a linear transformation associated with the matrix $A$, then
  \[ \text{range}(T_A) = \text{range}(A) \]

• **Nullspace of A Linear Transformation**

Let $V$ and $W$ be vector spaces and let $T : V \rightarrow W$ be a linear transformation. The nullspace of $T$, denoted by nullspace($T$), consists of all vectors $v$ in $V$, such that $T(v) = 0_W$.

- The nullspace($T$) is always a subspace of $V$ (can you prove it).
- If $T_A$ is a linear transformation associated with the matrix $A$, then
  \[ \text{nullspace}(T_A) = \text{nullspace}(A) \]

• **Examples**

3. For the linear transformation $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ given by

\[
T \left( \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ w - y \\ z \end{bmatrix}, \text{ give 2 examples of vectors in nullspace}(T). \]
• **Differentiation Operator**

Recall $\mathcal{P}_n$ be the vector space of all polynomials of degree at most $n$. Let $T = \frac{d}{dx}$ be the derivative operator. So

$$T(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = \frac{d}{dx}(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)$$

$$= a_1 + 2a_2 + \cdots + (n)a_n x^{n-1}$$

• **Example**

4. Show the differentiation operator $\frac{d}{dx} : \mathcal{P}_n \rightarrow \mathcal{P}_{n-1}$ is a linear transformation.

5. Find the nullspace of the linear transform $\frac{d}{dx}$.

• **Integration Operator** (from $\mathcal{P}_n \rightarrow \mathcal{P}_{n+1}$) is also a linear transformation.