5.3 Basis and Dimension

- **Basis**
  
  Let \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \) be vectors in a vector space \( V \). We define the set of vectors \( \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \} \) to be a *basis* for the vector space \( V \) if they satisfy the following two conditions:

  - (i) The vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \) are linearly independent.
  
  - (ii) Every vector in \( V \) can be written as a linear combination of the vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \). (in other words \( \text{span}(\{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \}) = V \))

- We're usually concerned about vector spaces which have finitely many vectors in a basis. They are called *finite dimensional vector spaces*.

- Vector spaces like \( C[a, b] \) (continuous functions with domain \( [a, b] \) are *infinite dimensional vector spaces*.

- **Examples**

  1. Prove \( \{ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \} \) is a basis for \( \mathbb{R}^3 \).

     Recall \( \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \).

  2. For \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \),

     Is \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) is a basis for \( \mathbb{R}^3 \)? Prove or disprove.
• **Standard Bases:**

  - **Standard Basis for** $\mathbb{R}^n$:
    
    is $\{e_1, e_2, \ldots, e_n\}$ where
    
    $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, ..., $e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

  - **Standard Basis for** $\mathcal{P}_n$:
    
    is $\{1, x, x^2, \ldots, x^n\}$

• **Example**

  3. Give a basis for $M_{2 \times 3}(\mathbb{R})$.

  4. Give a basis for the vector subspace $V = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ such that $w = 3z - z$

• **Theorem:** Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for the vector space $V$. Then any vector $u$ in $V$ can be uniquely expressed as a linear combination of the basis vectors.

  - In other words there are unique scalars $c_1, c_2, \ldots, c_k$ such that
    
    $u = c_1v_1 + c_2v_2 + \cdots + c_kv_k$.

• **Example**

  5. Prove the above Theorem.
• Example

6. We saw that \[
\begin{bmatrix}
1 & 0 & -1 \\
1 & 2 & 0 \\
0 & -1 & 1
\end{bmatrix}
\] is a basis for \(\mathbb{R}^3\).

Write the vector \[
\begin{bmatrix}
2 \\
-3 \\
1
\end{bmatrix}
\] as a linear combination of these vectors.

• **Thm:** If \(\{v_1, v_2, \ldots, v_k\}\) is a basis for \(V\), than any collection of \(k + 1\) (or more) vectors in \(V\) is linearly dependent.

• **Thm:** If a vector space \(V\) has a basis consisting of finitely many elements, then any two bases for \(V\) must contain the same number of elements.
- **Dimension**
  - A vector space $V$ is called *finite dimensional* if it has a basis consisting of a finite number of elements.
  - The unique number of elements in each basis of $V$ is called the **dimension** of $V$, written as $\dim(V)$.
  - If $V$ does not have a finite basis, then $V$ is said to be *infinite dimensional*.

- **Thm:** Let $V$ be a vector space and let $\dim(V) = n$. Then any set of $n$ linearly independent vectors must be a basis for $V$.

- **Example**
  7. What is the dimension of $\mathbb{R}^3$?
  8. What is the dimension of $\mathcal{P}_3$?
  9. What is the dimension of $\mathcal{M}_{m\times n}(\mathbb{R})$?
  10. Consider the trivial subspace of $\mathbb{R}^4$, $V = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$, what is $\dim(V)$?
  11. Which of the following sets form a basis for $\mathbb{R}^3$?

(a) $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} , \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} , \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} , \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} , \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \right\}$
How to picture a Basis

• The Standard Basis for \( \mathbb{R}^2 \)

\[ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ in Green and } e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ in Blue} \]

When we think of the vector \( \mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \) (in pink)

we know \( \begin{bmatrix} 3 \\ -1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) this linear combination looks like
• A Different Basis for $\mathbb{R}^2$: $e_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ in Green and $e_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in Blue

When we think of the vector $v = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ (in pink)

we know $\begin{bmatrix} 3 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ this linear combination looks like
• Yet Another Basis for $\mathbb{R}^2$:

$e_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ in Green and $e_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in Blue

When we think of the vector $v = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ (in pink)
we know \[
\begin{bmatrix}
3 \\
-1
\end{bmatrix}
= -1 \begin{bmatrix}
1 \\
3
\end{bmatrix} + 2 \begin{bmatrix}
2 \\
1
\end{bmatrix}
\] this linear combination looks like
• What if you tried to use linearly dependent vectors?

\[ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ in Green, } e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ in Blue, } v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ in Orange} \]

Give 3 different ways to write \( \begin{bmatrix} 3 \\ -1 \end{bmatrix} \) as a lin. comb. of \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), and \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \)

• What if you tried to use vectors that don’t span?

\[ v_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \text{ in Green and } v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ in Blue} \]

Try to write \( \begin{bmatrix} 3 \\ -1 \end{bmatrix} \) as a linear combination of \( \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \), and \( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \)

• These are why we require a Basis to both Linearly Independent and Spanning.