3.4 Elementary Matrices and Matrix Inverse

- A \( n \times n \) **elementary matrix** is a matrix which is obtained from the \( n \times n \) identity matrix \( I_{n \times n} \) by a single elementary row operation.

- **Elementary Row Operations**
  - **Interchanging the \( i \)-th and \( j \)-th rows of a matrix.**
    Denoted \( R_i \leftrightarrow R_j \)
  - **Multiplying a row by an non-zero scalar.**
    \( R_i \rightarrow cR_i \) denotes multiplying Row \( i \) by the scalar \( c \)
  - **Adding a multiple of one row to another**
    \( R_j \rightarrow R_j + cR_i \) denotes when he \( i \)-th row is multiplied by \( c \) and the result is added to the \( j \)-th row.

- **Examples** Which of the following are elementary matrices?

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>-3</td>
<td>0</td>
</tr>
</tbody>
</table>

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>-3</td>
<td>0</td>
</tr>
</tbody>
</table>

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>-3</td>
</tr>
</tbody>
</table>
• **Example**

6. Consider the Elementary Matrix \( E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

(a) What elementary row operation did this elementary matrix come from?

(b) Multiply \( EA \) for \( A = \begin{bmatrix} 1 & -2 & 0 & 4 \\ -2 & 3 & 1 & 0 \\ 6 & 7 & 8 & 9 \end{bmatrix} \)

(c) Now, do the row operation from part (a) to matrix \( A \).

• **Elementary matrices and row operations**

Let \( E \) be an elementary matrix, that is obtained from \( I_{n \times n} \) by an elementary row operation. Let \( A \) be any \( n \times n \) matrix. Then \( EA \) is the matrix that is obtained from \( A \) by applying the same elementary row operation.

• **Inverse matrix**

Let \( A \) be a square matrix of size \( n \times n \). We say that the matrix \( A \) is invertible or *non-singular* if there exists a \( n \times n \) matrix \( B \) such that \( AB = BA = I_{n \times n} \)

In such a case the matrix \( B \) is called the *inverse of the matrix \( A \).*

• **Example** Determine if either of the following matrices is an inverse for the matrix \( A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} \)

7. \( B = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix} \)

8. \( B = \begin{bmatrix} 3 & -1 \\ -1 & 0.5 \end{bmatrix} \)

• **Thm** If \( A \) is invertible, then the inverse of \( A \) is unique.
• **Properties of Matrix Inverse**
  
  – If $A$ is invertible, then so is $A^{-1}$, and
  
  $$(A^{-1})^{-1} = A$$
  
  – If $A$ is invertible, and if $c \neq 0$ is any scalar, then
  
  $$(cA)^{-1} = \frac{1}{c}A^{-1}$$
  
  – If $A$ and $B$ are invertible $n \times n$ matrices, then so is the product $AB$. Furthermore,
  
  $$(AB)^{-1} = B^{-1}A^{-1}$$
  
  – If $A$ is invertible, then so is $A^t$ and
  
  $$(A^t)^{-1} = (A^{-1})^t$$
  
• **Example**

  9. Simplify $(A^{-1}B)^{-1}(A^{-1}B^{-1}A^2)^{-1}$
  
  10. Simplify $A(AB)^t(AB^2A^t)^{-1}A$

• **Invertible Matrices and Systems of linear equations**

  If $A$ is invertible then the system of equations $Ax = b$ has a unique solution given by $x = A^{-1}b$

• **Examples** Solve the given systems of linear equations. You may want to use question 7 or 8 to help you.

  11. \[
  \begin{bmatrix}
  1 & 2 \\
  2 & 6
  \end{bmatrix}
  \begin{bmatrix}
  x \\
  y
  \end{bmatrix}
  =
  \begin{bmatrix}
  1 \\
  1
  \end{bmatrix}
  \]

  12. \[
  \begin{bmatrix}
  1 & 2 \\
  2 & 6
  \end{bmatrix}
  \begin{bmatrix}
  x \\
  y
  \end{bmatrix}
  =
  \begin{bmatrix}
  -2 \\
  4
  \end{bmatrix}
  \]
• **Left and Right Inverses**
  
  – We say that a $m \times n$ matrix $A$ (not necessarily square) has a **left inverse** if there exists an $n \times m$ matrix $B$ such that $BA = I_{n \times n}$
  
  – Similarly a $m \times n$ matrix $A$ (not necessarily square) has a **right inverse** if there exists a $n \times m$ matrix $C$ such that $AC = I_{m \times m}$

• **Thm** If a matrix is square, either a left or right inverse will be an inverse

• **Theorem:** Let $A$ be an $n \times n$ matrix. Then the following statements are equivalent.
  
  – (1) There exists a $n \times n$ matrix $B$ such that $BA = I_{n \times n}$, i.e. the matrix $A$ has a left inverse $B$.
  
  – (2) The system of homogeneous equations $Ax = 0$ has only the trivial solution $x = 0$.
  
  – (3) The matrix $A$ can be row reduced to an echelon matrix with $n$ pivots, i.e. the rank (number of pivot elements) of $A$ is $n$.
  
  – (4) The system of equations $Ax = b$ has a solution (is consistent) for all vectors $b$ in $\mathbb{R}^n$. For each such $b$, the solution vector is unique.
  
  – (5) The matrix $A$ has a right inverse.
  
  – (6) The matrix $A$ is invertible, i.e. there exists a unique matrix $A^{-1}$ such that $AA^{-1} = A^{-1}A = I$.

• **Examples:** Note that the matrix $B = \begin{bmatrix} 3 & -1 \\ -1 & 0.5 \end{bmatrix}$ is from example 8

  13. How many solutions does $\begin{bmatrix} 3 & -1 \\ -1 & 0.5 \end{bmatrix} x = \begin{bmatrix} 1 \\ -7 \end{bmatrix}$ have?

  14. How many solutions does $\begin{bmatrix} 3 & -1 \\ -1 & 0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.931 \\ 3.853 \end{bmatrix}$ have?

  15. Find all solutions to $\begin{bmatrix} 3 & -1 \\ -1 & 0.5 \end{bmatrix} x = 0$.

  16. How many pivots are in an REF of $\begin{bmatrix} 3 & -1 \\ -1 & 0.5 \end{bmatrix}$?
Finding the Inverse of a Matrix

- If $A$ is an $n \times n$ matrix whose RREF is $I_n$, let $E_1, E_2, \ldots, E_n$ be the matrices corresponding to the row operations that reduce $A$ to the identity

$$E_k \cdots E_2 E_1 A = I$$

- So $(E_k \cdots E_2 E_1)A = I$, therefore $(E_k \cdots E_2 E_1) = A^{-1}$
- Or $(E_k \cdots E_2 E_1)I = A^{-1}$
- *All we need to do to find $A^{-1}$ is do the exact same row operations on $I$ that we did to $A$ to reduce it to $I_n$*

Finding the Inverse of $A$

- For $A$ an $n \times n$ matrix.
- Create a augmented matrix $[A | I_n]$
- Reduce this matrix to RREF
  * IF the left side has $n$ pivots (i.e. if the left side is $I_n$), then $A$ is invertible and the right hand side is $A^{-1}$.
  * IF the left side has less than $n$ pivots (isn’t $I_n$), then $A$ is not invertible.

Vocabulary:

For $A$ an $n \times n$ matrix,
- If $A^{-1}$ does not exist, we say $A$ is singular (i.e. not-invertible)
- If $A^{-1}$ exists, we say $A$ is non-singular

Examples: Find the inverse of the given matrix, if it exists.

17. $A = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$
18. $A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$
19. $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 3 & 0 \end{bmatrix}$
20. (optional) $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ -1 & -2 & 0 & 1 \end{bmatrix}$
• **Example:** *Start/set up* the following.

21. For \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \), and \( B = \begin{bmatrix} -3 & 4 \\ 5 & -18 \end{bmatrix} \), find all \( 2 \times 2 \) matrices \( X \) satisfying:

\[
AX + 2X = B
\]