2.1 Row echelon form of a matrix

- System of \( m \) linear equations in \( n \) variables (unknowns)

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

where the coefficients \( a_{ij} \) are real numbers.

- Using Matrix Notation

\[
Ax = b
\]

or

\[
\begin{bmatrix}
    a_{11} & a_{12} & + \cdots + a_{1n} \\
    a_{21} & a_{22} & + \cdots + a_{2n} \\
    \vdots & \vdots & \vdots \\
    a_{m1} & a_{m2} & + \cdots + a_{mn}
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{bmatrix}
= 
\begin{bmatrix}
    b_1 \\
    b_2 \\
    \vdots \\
    b_n
\end{bmatrix}
\]

- we’ll often use \( x = \begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{bmatrix} \) and \( b = \begin{bmatrix}
    b_1 \\
    b_2 \\
    \vdots \\
    b_n
\end{bmatrix} \)

- we call \( A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \vdots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix} \), the Matrix of Coefficients of the system of linear equations

- Augmented Matrix

\[
[A|b] = 
\begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
    a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
    \vdots & \vdots & \cdots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn} & b_n
\end{bmatrix}
\]
• **Example**

1. Write the augmented matrix associated with the system of equations:

   (a) \[
   \begin{align*}
   3x - 100y + 4z &= 6 \\
   -9.2x + 11z &= 17
   \end{align*}
   \]

   \[
   \begin{align*}
   -3x_1 - 3x_2 + \frac{2}{5}x_3 + x_4 &= 0 \\
   7x_1 + 1x_2 - x_3 &= 10 \\
   \frac{1}{2}x_1 + 7x_2 + 2x_3 + 6x_4 &= -\frac{3}{8}
   \end{align*}
   \]

   *Note: you were NOT asked to solve this system...*

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• **Solving a System of Linear Equations:** The system remains unchanged if we:

  – Switch the position of any 2 rows
  – Multiply any equation by a non-zero constant
  – Take a multiple of one equation and add it to another

• **Elementary Row Operations:**

  – **Interchanging the \( i \)-th and \( j \)-th rows of a matrix.**
    Denoted \( R_i \leftrightarrow R_j \)

  – **Multiplying a row by an non-zero scalar.**
    \( R_i \rightarrow cR_i \) denotes multiplying Row \( i \) by the scalar \( c \)

  – **Adding a multiple of one row to another**
    \( R_j \rightarrow R_j + cR_i \) denotes when he \( i \)-th row is multiplied by \( c \) and the result is added to the \( j \)-th row.

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• A matrix \( A \) is **row equivalent** to a matrix \( B \), if \( B \) can be obtained from \( A \) by elementary row operations.

• Let \( Ax = b \) and \( A'x = b' \) be two systems of linear equations. If the augmented matrices \([A|b]\) and \([A'|b']\) are row equivalent, then the two systems \( Ax = b \) and \( A'x = b' \) have the same solution vectors.
• **Row Echelon Form** A matrix $A$ is in row echelon form if the following conditions are satisfied.

  – All rows consisting entirely of zeros (if any) appear at the bottom of $A$.
  – The first (from the left) non-zero entry of a non-zero row is called a *pivot*. In two successive rows the pivot entry of the lower row occurs to the right of the pivot entry for the higher row.

• **Example:**

  2. Which of the following matrices are in Row Echelon Form?

    (a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$

    (b) $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

    (c) $\begin{bmatrix} 0 & 0 & 0 \\ 5 & \frac{2}{5} & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

    (d) $\begin{bmatrix} -3 & 4 & 3.2 & 0 & \pi & 2 \\ 0 & 0 & 2 & -1 & 0 & 11 \\ 0 & 0 & 0 & 7 & 7.7 & 7.77 \end{bmatrix}$

    (e) $\begin{bmatrix} 0 & -2 & 4 \\ 0 & 0 & 23 \\ 0 & 0 & 0 \end{bmatrix}$

  3. Use row operations to find a Row Echelon matrix that is Row Equivalent to the matrix $M = \begin{bmatrix} 0 & 1 & 1 & 3 \\ 1 & 2 & 1 & 0 \\ 2 & -2 & 0 & -2 \end{bmatrix}$
- **Gaussian Elimination**: An algorithm for transforming any matrix into a Row Echelon Form that is row equivalent to the original matrix.
  
  - Step 1: Locate the first (leftmost) pivot column (non-zero column). If necessary, interchange rows so that a Pivot is on top of that column.
  - Step 2: (optional) Normalize so the pivot is 1
  - Step 3: Zero out all entries in the pivot column below the pivot entry. So this by adding suitable multiples of the pivot row to each row below
  - Step 4: Mentally ignore the pivot row(s) on top of the matrix, and proceed to Step 1 for the remaining matrix.

- **Example**

  4. Use row operations to find a Row Echelon matrix that is Row Equivalent to the matrix
     \[
     \begin{bmatrix}
     2 & -1 & 4 \\
     4 & -1 & 8 \\
     -1 & 1 & -2 \\
     \end{bmatrix}
     \]
     
     *Hint: Gaussian Elimination helps you do this efficiently*

  5. Use row operations to find a Row Echelon matrix that is Row Equivalent to the matrix
     \[
     \begin{bmatrix}
     3 & 0 & -3 & 0 & 1 \\
     1 & 2 & -1 & -2 & 3 \\
     -6 & 0 & 6 & 2 & -4 \\
     \end{bmatrix}
     \]
• **True/False?** and explain why.

  – If matrix $A$ is row equivalent to matrix $B$ then $B$ is row equivalent to $A$.

  – If $A$ is row equivalent to matrix $B$ and $B$ is row equivalent to Matrix $C$, then $A$ is row equivalent to $C$.

  – The Row Echelon form of a matrix is unique.

• **Thm:** Let $A$ and $B$ be two row echelon matrices which are row equivalent to each other. Then the $i$-th column of $A$ is a pivot column if and only if the $i$-th column of $B$ is also a pivot column.

  In other words, if two matrices are row equivalent, they will have exactly the same pivot columns.