Review for Midterm 2

Set theory

**Basics (C-level)**
- set, function \( f : A \to B \), domain, codomain, \( f \circ g \)
- one-to-one function \( \forall x, y \in A : f(x) = f(y) \implies x = y \)
- onto function: \( \forall y \in B \exists x \in A : f(x) = y \)
- Be able to prove that, e.g., injective function composed with injective function is injective.

**Cardinality (B-level)**
- \( A \sim B \) exists \( f : A \to B \) which is 1-1 and onto
- \( A \leq B \) exists \( f : A \to B \) which is 1-1
- Countable \( A \leq \mathbb{N} \)
- Basic facts about countable sets
- Statement of Cantor-Schöder-Bernstein:
  \( (A \leq B \text{ and } B \leq A) \text{ then } A \sim B \)

**Diagonalization (A-level)**
- Proof that \( \mathbb{N} \not\sim [0,1]^{\infty} \)
- Proof that \( A \not\sim \mathcal{P}(A) \)

**Effectivity (B-level)**
- Decidable set (can decide if element is in/out)
- Effectively enumerable (can list elements)
- Semidecidable (can decide if element is in)
- Basic properties of these definitions
First-order logic

Basics (C-level)

* Be able to translate English into first-order logic using
  \( A, \exists, \iff, \neg, \land, \lor, \forall, \exists, = \)
  (Can write naturally, e.g. \( x + y < z \))

* Translate first-order logic into English.

* Common idea
  "All apples are bad" (\( A \) means all things)
  \( \forall x (Ax \to Bx) \)
  \( \iff \) for \( \forall \) use \( \to \)
  "Exists a bad apple"
  \( \exists x (Ax \land Bx) \)
  \( \iff \) for \( \exists \) use \( \land \)

* Be able to label free variables

* Be able to recognize
  * functions, constants
  * predicates
  * terms
  * atomic formulas
  * well-formed formulas
  * sentences

\( f, c \)  \( P \)  \( f, g, c \)  \( x, y, z \)  \( \text{constant, } P \text{ with } f, g, c \text{ w/ no free variables} \)
Formal inductive/recursive definitions (B-level)

**Symbols**
- Predicate symbols (n-place)
- Function symbols (n-place)
- Constant symbols = 0-place Function symbols
- Variables \( v_i \)
- Logic connectives \( \land, \lor, \rightarrow \) like a 0-place predicate symbol
- Equality \( = \) like a 2-place predicate
- Quantifier \( \forall \) Symbol

**Terms**
- Variable \( v_i \)
- \( \emptyset \) (Base) \( v_i \) is a term
- \( \{ t_1, \ldots, t_n \} \) is a term
- \( n \)-place Function symbol
- Constant \( c \)
- \( \emptyset \) ' \( c \) is a term (follows from \( \emptyset \))

- Be able to do induction/recursion on terms.

**Atomic formulas**
- \( \emptyset \) \( P(t_1, \ldots, t_n) \) is atomic formula (follows from \( \emptyset \))
- \( \emptyset \) ' \( t_1 = t_2 \) is atomic (follows from \( \emptyset \))
- \( \emptyset \) '' \( \bot \) is atomic (follows from \( \emptyset \))
WFFs

1. Atomic formulas are WFFs
2. \((\alpha \rightarrow \beta)\) are WFFs
3. \(\forall x \alpha\) are WFFs

- Be able to do induction/recursion on WFFs

Sentences
WFFs with no free variables

Structures and logical implication (\(\mathcal{L}\)-level)

Structure \(\mathcal{O}\)
1. \(10 \subseteq\) set of objects (universe)
2. \(\sigma: 10 \rightarrow 10\) function
3. \(p: 10 \times 10\) relation
4. \(c: 10\) constant

Term Evaluation

Setup: \(\mathcal{O}\) structure, \(s: V \rightarrow 10\)

- \(s(v) = s(x)\)
- \(s(f(t_1, \ldots, t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n))\)
- \(s(c) = c\)
For $\varphi \in \text{ESJ}$ (OR satisfies $\varphi$ with $s$)

6. For $P \leq_1 \ldots \leq_n \in \text{ESJ}$ \iff $\langle s(t_1), \ldots, s(t_n) \rangle \in P$

7. $\forall_0 (t_1 = t_2) \in \text{ESJ}$ \iff $s(t_1) = s(t_2)$

8. $\forall \omega \in \text{ESJ}$

1. $\models (\alpha \rightarrow \beta) \in \text{ESJ}$ \iff $(\forall \alpha \in \text{ESJ} \implies \models \beta \in \text{ESJ})$ \iff $(\forall \alpha \in \text{ESJ} \lor \models \beta \in \text{ESJ})$

2. $\models \forall x \in \text{ESJ}$ \iff for all $d \in \text{LO}$

$\models \forall x \in \text{S}(x^{1d})$

$s$ except $s(x) = d$

**Other first-order stuff (B-level)**

Thm If $s, s' : \nu \rightarrow \text{LO}$ agree on $\sqrt{\varphi}$, then

$\models \varphi \in \text{ESJ}$ \iff $\models \varphi \in \text{ESJ}'$

For $\varphi [a_1, \ldots, a_n] \iff \models \varphi \in \text{ESJ}$

where $s(v_i) = a_i$

sentence

$\mathcal{O}$ is a model of $\sigma$ \iff $\models \sigma$

• $\Gamma \models \varphi$ ( $\Gamma$ tautologically implies $\varphi$) if for all $\mathcal{O}, s$ if $\models \varphi \in \text{ESJ}$ for all $y \in \Gamma$ then $\models \varphi \in \text{ESJ}$

• $\varphi$ is valid \iff $\models \varphi$
Induction of proofs (A-level)

- Induction on proofs
- Statement of Soundness Theorem
- Completeness Theorem
- Main ideas in proof of completeness theorem