Effectiveness (1.7)

Question
Given $\Sigma$ and $\tau$, can we "decide" whether $\Sigma \models \tau$ (or $\Sigma \not\models \tau$) using an "effective procedure"?

Informally, an effective procedure must meet the following conditions:

1. It is given by a finite set of instructions. (Think computer program or instructions for a human assistant who doesn't know math)

2. They must be mechanically implemented. (No intelligence is assumed on the part of the machine/person following the instructions. Also, no randomness, e.g., coin-flipping, is used.)

3. After a finite number of steps it produces yes/no.

Note: There is no upper bound on
- the # of steps
- the size of the instructions
- the amount of time it takes
- the amount of resources (e.g., computer memory)
Thm 17B
There is an effective procedure for deciding whether an expression is a wff.

Proof See algorithm in §1.3.

Remk Since we have countably infinitely many sentence symbols, we can replace, say, $A_6$ with $A^{'''''}$. Then we have only the symbols ($, )$, $\land$, $\lor$, $\rightarrow$, $\leftrightarrow$, $A$, $'$, which we can identify with $0-9$.

Def A set of expressions $\Sigma$ is decidable if there is an effective procedure that, given an expression $\alpha$, will decide whether or not $\alpha \in \Sigma$.

Example: The set of wffs is decidable. The set of wffs only containing $A$, $\rightarrow$, ($, )$, $\land$, $'$ is decidable.

Thm 17C
There is an effective procedure that, given a finite set $\Sigma$ of wffs and a wff $\alpha$, will decide whether or not $\Sigma \models \alpha$.

Proof Use truth table.

Corollary Given finite $\Sigma$, the set $\{ \alpha \mid \alpha \text{ wff, } \Sigma \models \alpha \}$ is decidable.

Proof Use a truth table.
Remark If $\Sigma$ is infinite, then the set of tautological consequences may not be decidable.

Def A set $A$ of expressions is effectively enumerable iff there is an effective procedure that lists (in some order) the members of $A$.

Def A set $A$ of expressions is semidecidable iff there exists an effective procedure that, given any expression $\varepsilon$, produces the answer "yes" if $\varepsilon \in A$.

Thm 17E A set $A$ is effectively enumerable iff it is semidecidable.

Proof
\((\Rightarrow)\) Assume $A$ is effectively enumerable. Fix $\varepsilon$. Wait for our effective procedure to output $\varepsilon$. When it does, output "yes." (If it doesn't, then never output anything.)

\((\Leftarrow)\) Assume $A$ is semidecidable. We want to use the "semi-decision procedure" to create a list $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ of the elements of $A$. 
Do the following:

- Spend 1 minute (say 10^10 steps) checking if \( E_1 \in A \)
- Spend 2 minutes each checking if \( E_1 \in A \) and \( E_2 \in A \),
- Spend 3 minutes each checking if \( E_1 \in A, E_2 \in A, E_3 \in A \)

Whenever we find some \( E \in A \), add it to our list if we haven't already added it. If \( E \in A \) we will eventually find it since \( A \) is semidecidable.

**Theorem 17.5**

A set of expressions are decidable if both it and its complement are effectively enumerable (semidecidable).

**Proof** If \( A \) is decidable, it is semidecidable.

If \( A \) is semidecidable and its complement \( A^c \) is semidecidable, then run both "semidecision procedures" until one of them returns "yes".

If \( E \notin A \), the first will return "yes.
If \( E \in A \), then second \quad \square \quad \square
Thm 17.6
If $\Sigma$ is a decidable set of wffs (or even an effectively enumerable set) then the set of tautological implications of $\Sigma$ is effectively enumerable.

Proof List out all the wffs in $\Sigma$

$\sigma_1, \sigma_2, \sigma_3$

For each wff $\tau$, test whether

$\Sigma \vdash \tau$

$\sigma_1 \vdash \tau$

$\sigma_1, \sigma_2 \vdash \tau$

If one of these holds, then we know $\Sigma \vdash \tau$ and we put $\tau$ into our list.

If none of these hold, then by the compactness theorem, $\Sigma \not\vdash \tau$.
(Recall, if $\Sigma \vdash \tau$ then there is a finite $\Sigma_0 \subseteq \Sigma$ s.t. $\Sigma_0 \vdash \tau$. )