Cardinality

Cardinality is the "size" of a set.

**Def.** $A \sim B$ (sets $A$ and $B$ are equinumerous) iff there is a one-to-one and onto (a.k.a. bijective) function $\varphi: A \to B$. Then say $\text{card } A = \text{card } B$.

**Def.** A set $A$ is finite iff $A \sim \{1, \ldots, n\}$ for some natural number $n$.

Say $\text{card } A = n$. In other cases, $A$ is infinite.

**Examples.** $\text{card } \emptyset = 0$, $\text{card } \{\sqrt{2}, \pi\} = 2$.

**Def.** A set $A$ is countable iff there is a one-to-one map $\varphi: A \to \mathbb{N}$.

**Theorem.** A set $A$ is countable iff either

- 1. $A$ is finite, or
- 2. $A \sim \mathbb{N}$.

**Proof.** ($\Rightarrow$) If $A$ is finite, then there is a one-to-one (and onto) function $\varphi: A \to \{1, \ldots, n\}$. This is also a one-to-one function (no longer onto) $\varphi: A \to \mathbb{N}$.

If $A \sim \mathbb{N}$, there is a one-to-one (and onto) function $\varphi: A \to \mathbb{N}$.
(\Rightarrow) If \( A \) is countable, there is a one-to-one function \( f: A \to \mathbb{N} \).
If \( f \) is finite, we are done.
So assume \( f \) is infinite.
We want to construct a new function \( g: A \to \mathbb{N} \) which is one-to-one and onto.

Let \( n_0 \in \mathbb{N} \) be the least element in the range of \( f \). Since \( n_0 \in \text{ran } f \), there is some \( a_0 \in A \) s.t. \( f(a_0) = n_0 \).
Since \( f \) is one-to-one, this \( a_0 \) is unique.
Set \( g(a_0) = 0 \).

Let \( n_1 \in \mathbb{N} \) be the next least in ran \( f \).
Let \( a_1 \in A \) s.t. \( f(a_1) = n_1 \). Set \( g(a_1) = 1 \).

Since \( A \) is infinite we can keep doing this for all \( i \in \mathbb{N} \) to get \( a_i \in A \) and
set \( g(a_i) = i \).

Remark: Now we see that "countable" means we can "count" or "enumerate" all elements
of \( A \). That is, \( A \subseteq \{a_0, a_1, a_2, a_3, \ldots \} \).
The following sets are countable:

1. \( \mathbb{N} \)
2. \( \mathbb{Z} \)
3. \( \mathbb{Q} \)
4. Set of algebraic numbers (solutions to polynomials
   \( a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0 \) where
   \( a_n, a_{n-1}, \ldots, a_0 \in \mathbb{Z} \))
5. A countable union of countable sets.
6. \( A^n = A \times \cdots \times A \) (the set of \( n \)-tuples)
   assuming \( A \) is countable.
7. A subset of a countable set.
8. The set of all finite sequences
   of \( A \), where \( A \) is countable.
9. Every set \( A \) for which there is an
   onto function \( f : \mathbb{N} \to A \).
10. The set of well-formed countably
    many sentence symbols,
Thm ~ is an equivalence relation.
(HW Exercise)

Def: $A \leq B$ if there is a one-to-one function $g: A \rightarrow B$. We then say $\text{card } A \leq \text{card } B$

Thm (Cantor–Schröder–Bernstein):
If $A \leq B$ and $B \leq A$, then $A \sim B$.

Proof sketch:
We have one-to-one maps $f: A \rightarrow B$ and $g: B \rightarrow A$.
Then we can partition $A$ and $B$ as follows.

To get a one-to-one and onto map $h: A \rightarrow B$, use the following:

$h(x) = \begin{cases} f(x) & \text{if } x \text{ is in the shaded region} \\ g^{-1}(x) & \text{if } x \text{ is in the white region} \\ f(x) & \text{otherwise} \end{cases}$

You can check that $h$ is one-to-one and onto.

Thm (Requires axiom of choice):
For all sets $A$ and $B$, either $A \leq B$ or $B \leq A$. 