Compactness (§1.7)

Def. A set of wffs $\Sigma$ is *satisfiable* if there is a truth assignment that satisfies every member of $\Sigma$.

A set of wffs $\Sigma$ is *finitely satisfiable* if every finite subset $\Sigma_0 \subseteq \Sigma$ is satisfiable.

Compactness Theorem (version 1)

A set of wffs is satisfiable iff it is finitely satisfiable.

Proof.

(⇒) Straightforward by def of satisfiable.

(⇐) Assume $\Sigma$ is finitely satisfiable. Our proof has two parts.

1. Extend $\Sigma$ to a maximal finitely satisfiable set $\Delta$.

2. Use $\Delta$ to find a truth assignment satisfying $\Sigma$.?
Part 1
Let \( \alpha_1, \alpha_2, \ldots \) be a fixed enumeration of the wffs.

This is possible since we have countably many symbols. For example, we can first list the formulas of length 1 that use only \( \alpha_1 \):

\[ \alpha_1 \]

Then list the formulas of length \( \leq 2 \) that use only \( \alpha_1, \alpha_2 \) (and haven’t already been listed):

\[ \alpha_2 \]

Then for 3 and 4 we have:

\[ \alpha_3, (\neg \alpha_1), (\neg \alpha_2), (\neg \alpha_3) \]

For 5,

\[ \alpha_5, (\neg \alpha_5), (\alpha_1 \land \alpha_1), (\alpha_1 \land \alpha_1), \ldots \]

Define \( \Delta_n \) recursively:

\[ \Delta_0 = \Sigma \]

\[ \Delta_{n+1} = \begin{cases} \Delta_n ; \text{and if this is finitely satisfiable} \\ \Delta_n ; \text{otherwise} \end{cases} \]

Claim: Each \( \Delta_n \) is finitely satisfiable.
Proof: HW exercise (#1 p 65) \( \square \)

Let \( \Delta = \bigcup \Delta_n \).

Three properties of \( \Delta \):

1. \( \Sigma \subseteq \Delta \)
2. For all wffs \( \alpha \), \( \alpha \in \Delta \) or \((\neg \alpha) \in \Delta \).
3. \( \Delta \) is finitely satisfiable.
Part 2
Let \( v \) be the truth assignment given by
\[
v(A_n) = T \text{ if } A_n \in \Delta
\]
for all sentence symbols \( A_n \).

By property (6) of \( \Delta \), \( A_n \in \Delta \) or \( \forall A_n \in \Delta \)

\[
\text{Claim: For all wff } \phi, \quad v \text{ satisfies } \phi \text{ iff } \phi \in \Delta.
\]

\[
\text{Proof: Homework problem (\#2, p. 65)}
\]
\[
\text{(Use induction on wffs) } \square
\]

Since \( \Sigma \subseteq \Delta \), \( v \) satisfies every member of \( \Sigma \). This finishes the proof of the compactness theorem. \( \square \)
Corollary 17A (Compactness Thm, version 2)

If $\Sigma \models \phi$, then there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \phi$.

Lemma $\Sigma \not\models \phi$ if $\Sigma \cup \phi$ is not satisfiable.

Proof We will show

$\Sigma \not\models \phi$ iff $\Sigma \cup \phi$ is satisfiable.

($\Rightarrow$) Assume $\Sigma \not\models \phi$. Then there is a truth assignment $\nu$ satisfying $\Sigma$ but not $\phi$. So $\nu$ satisfies $\phi$. So $\nu$ satisfies $\Sigma \cup \phi$.

($\Leftarrow$) Assume $\Sigma \cup \phi$ is satisfiable. Then there is a truth assignment $\nu$ satisfying $\Sigma$ and $\phi$. So $\nu$ does not satisfy $\phi$. So $\Sigma \not\models \phi$.

Proof of corollary

Prove the contrapositive.

$\Sigma_0 \not\models \phi$ for all finite $\Sigma_0 \subseteq \Sigma$

$\Rightarrow \Sigma_0 \cup \phi$ is satisfiable for all finite $\Sigma_0 \subseteq \Sigma$ (lemma)

$\Rightarrow \Sigma \cup \phi$ is finitely satisfiable

$\Rightarrow \Sigma \cup \phi$ is satisfiable (compactness thm)

$\Rightarrow \Sigma \not\models \phi$. (lemma)

Exercise Show the compactness theorem follows from the corollary.