A quick summary of §1.3 and §1.4

Recall we defined the wffs in a way that lets us do induction on them. Also, we were able to define functions, such as \( \overline{v}(\alpha) \) recursively.

However, why is \( \overline{v}(\alpha) \) unique? In other words, why can't I have \( \overline{v}_1 \) and \( \overline{v}_2 \) which agree on all the rules constructing \( \overline{v} \), for example:

1. \( \overline{v}(An) = v(An) \)
2. \( \overline{v}((\alpha \land \beta)) = \begin{cases} T & \text{if } \overline{v}(\alpha) = \overline{v}(\beta) = T \\ F & \text{otherwise.} \end{cases} \)

The short answer is that we can uniquely parse any formula into its tree. (See §1.3 for details.)

The long answer is that we have the following theorem:

**Unique readability theorem**

Every wff \( \phi \) is constructed in a unique way from other formulas. For example, for any \( \alpha, \beta, \gamma, \delta \)

\[ E\land(\alpha, \beta) \neq E\lor(\alpha, \beta). \]

Also, for any \( \alpha_1 \neq \alpha_2 \) and any \( \beta, \gamma \)

\[ E\Rightarrow(\alpha_1, \beta) \neq E\Rightarrow(\alpha_2, \gamma). \]

Etc.

(See §1.4 for a proof.)
**Polish notation**

There is another notation which does not use parentheses.

With **Polish notation**, one writes:

- \( \neg \alpha \) instead of \( \neg \alpha \)
- \( \alpha \land \beta \) instead of \( \alpha \land \beta \)
- \( \alpha \lor \beta \) instead of \( \alpha \lor \beta \)
- \( \alpha \rightarrow \beta \) instead of \( \alpha \rightarrow \beta \)
- \( \alpha \leftrightarrow \beta \) instead of \( \alpha \leftrightarrow \beta \)

(The connective goes first.)

The can easily parse as follows:

\[ \rightarrow \land A \lor \neg B \leftrightarrow C \land B \]

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(Complete the diagram with the appropriate symbols.)
Omitting Parentheses

These rules/abbreviations will be adopted:

1. Don't need to write outmost parentheses
   \[ A \lor B \text{ means } (A \lor B) \]

2. Negation symbol applies to as little as possible
   \[ \neg A \land B \text{ means } (\neg A) \land B \]
   
   \[ \text{not } ((\neg (A \land B)) \]

3. Conjunction and disjunction apply to as little as possible after rule 2.
   \[ A \land B \rightarrow \neg \lor D \text{ means } ((A \land B) \rightarrow (\neg C) \lor D)) \]

4. If \( \land, \lor, \rightarrow, \leftrightarrow \) are used in a row the group on the right
   \[ \alpha \land \beta \land \gamma \text{ means } \alpha \land (\beta \land \gamma) \]
   
   \[ \text{[although it is taut. equiv. to } (\alpha \land \beta) \land \gamma] \]

5. \[ \alpha \rightarrow \beta \rightarrow \gamma \text{ means } \alpha \rightarrow (\beta \rightarrow \gamma) \]
   
   \[ \text{Note: This is tautologically equiv. to } \]
   
   \[ (\alpha \land \beta) \rightarrow \gamma \]
   
   \[ \text{It is not taut. equiv. to either } \]
   
   \[ (\alpha \rightarrow \beta) \land (\beta \rightarrow \gamma) \]
   
   \[ (\alpha \rightarrow \beta) \rightarrow \gamma \]
Induction and recursion (quick summary)

One can define a set of objects inductively as follows:

1. Take a large set of objects \( U \).
2. Take a small subset of starting objects \( B \subseteq U \).
3. Let \( C \) be the smallest subset of \( U \) containing \( B \) that is closed under all operations in some class \( F \).

Examples

1. **Natural numbers** (\( \mathbb{N} \))
   
   \[
   U = \mathbb{N} \\
   B = \{0, 1\} \\
   F = \{S, \#\} \quad \text{where} \quad S(x) = x + 1 \\
   0 \in \mathbb{N} \\
   n \in \mathbb{N} \Rightarrow n + 1 \in \mathbb{N}
   \]

2. **Integers** (\( \mathbb{Z} \))
   
   \[
   U = \mathbb{Z} \\
   B = \{0, 1\} \\
   F = \{S, P\} \\
   S(x) = x + 1 \\
   P(x) = x - 1 \\
   0 \in \mathbb{Z} \\
   n \in \mathbb{Z} \Rightarrow n + 1 \in \mathbb{Z} \\
   n \in \mathbb{Z} \Rightarrow n - 1 \in \mathbb{Z}
   \]

3. **Positive rationals** (\( \mathbb{Q}^+ \))
   
   \[
   U = \mathbb{Q}^+ \\
   B = \{1\} \\
   F = \{S, \div\} \\
   S(x) = x + 1 \\
   D(x, y) = \frac{x}{y} \\
   1 \in \mathbb{Q}^+ \\
   x \in \mathbb{Q}^+ \Rightarrow x + 1 \in \mathbb{Q}^+ \\
   x, y \in \mathbb{Q}^+ \Rightarrow \frac{x}{y} \in \mathbb{Q}^+
   \]
∀ ws
U = set of expressions
B = set of sentence symbols
F = {E_7, E_8, E_9, E_10, E_11, E_12}

Induction
Anything defined this way you can
do induction on it. For example,
to prove a property P(x) holds for all x ∈ \mathbb{Q}^+
it is enough to show

• P(1) holds (base case)
• If P(x) holds, then P(x+1) holds
• If P(x), P(y) hold, then P(x/y) holds.

Recursion
We can also use recursion to define
functions on an inductively defined set.

Bad example
Let's define the "denominator" of x ∈ \mathbb{Q}^+
as follows:
\[
\begin{align*}
d(1) &= 2 \\
d(x+1) &= d(x) \\
n(1) &= 1 \\
n(x) &= n(x) + d(x) \\
\frac{\alpha}{b} + 1 &= \frac{\alpha + b}{b} \\
d(\frac{x}{y}) &= d(x)n(y) \\
n(\frac{x}{y}) &= n(x)d(y) \\
\frac{\alpha}{b} / \frac{c}{d} &= \frac{\alpha d}{bc}
\end{align*}
\]

Problem
\[
1 = \frac{1+1}{1+1} \quad \text{but}
\]
\[
n(1) = 1 \quad \text{and} \quad n\left(\frac{1+1}{1+1}\right) = 2
\]
So it is not well-defined.
In order for recursion to work, we need that our function is \emph{well-defined} if \( x = y \), then \( f(x) = f(y) \).

The last example is not well-defined.

However, recursion is always well-defined if our inductive set \( C \) is \emph{freely-generated}.

An inductive set \( C \) is \emph{freely generated} if "there is only one way to get to \( x \in C \) using \( B \) and \( g \)."

\( \mathbb{N} \), \( \mathbb{W} \)s are \emph{freely generated} (for \( \mathbb{W} \)s, this is the \emph{uniqueness readibility theorem}).

\( \mathbb{Z} \), \( \mathbb{Q}^+ \) are not \emph{freely generated}.

(See §1.4 for all the details.)