A list of important tautologies

1. ∧, ∨, ¬ are associative and commutative
   For example,
   \[ ((A \land (B \land C)) \iff ((A \land B) \land C)) \]
   \[ ((A \land B) \iff (B \land A)) \]

2. Distributive laws
   \[ (A \land (B \lor C)) \iff ((A \land B) \lor (A \land C)) \]
   \[ (A \lor (B \land C)) \iff ((A \lor B) \land (A \lor C)) \]

3. Negation
   \[ (\neg (\neg A)) \iff A \]
   \[ (\neg (A \rightarrow B)) \iff (A \land \neg B) \]
   \[ (\neg (A \leftrightarrow B)) \iff ((A \land \neg B) \lor (\neg A) \land B) \]

4. De Morgan's laws
   \[ (\neg (A \land B)) \iff ((\neg A) \lor (\neg B)) \]
   \[ (\neg (A \lor B)) \iff ((\neg A) \land (\neg B)) \]

5. Excluded middle
   \[ (A \lor (\neg A)) \]
   Contradiction \[ (\neg (A \land (\neg A))) \]
   Contraposition \[ (((A \rightarrow B) \iff (\neg B) \rightarrow (\neg A))) \]
   Exportation \[ (((A \land B) \rightarrow C) \iff (A \rightarrow (B \rightarrow C))) \]
Some basic properties of $\Sigma \vdash \alpha \iff \Sigma ; \alpha \vdash \beta$

**Theorem:** If $\Sigma \vdash \alpha$ and $\Sigma ; \alpha \vdash \beta$, then $\Sigma \vdash \beta$.

**Remark:** The definition of $\Sigma \vdash \beta$ technically requires that we use truth assignments $\nu : S \rightarrow \{T, F\}$ where $S$ is exactly the sentence symbols in $\Sigma$ and $\beta$. This is a problem since $\alpha$ may contain additional symbols.

Lemme (Exercise #6 on p. 27)
Let $S$ be a set of sentence symbols which contain all those in $\Sigma$ and $\beta$ (and possibly more). Then $\Sigma \vdash \beta$ if and only if for all $\nu : S \rightarrow \{T, F\}$, if for all $\gamma \in \Sigma$, $\nu(\gamma) = T$, then $\nu(\beta) = T$.

**Proof of Theorem**
Assume $\Sigma \vdash \alpha$ and $\Sigma ; \alpha \vdash \beta$.
We want to show $\Sigma \vdash \beta$.
Let $S$ be the sentence symbols in $\Sigma, \alpha, \beta$.
Let $\nu : S \rightarrow \{T, F\}$ be an arbitrary truth assignment. Assume for all $\gamma \in \Sigma$ that $\nu(\gamma) = T$.
By the lemma, it is enough to show that $\nu(\beta) = T$.
Since $\Sigma \vdash \alpha$, we have that $\nu(\alpha) = T$.
So $\nu(\gamma) = T$ for all $\gamma \in \Sigma ; \alpha$.
So, since $\Sigma ; \alpha \vdash \beta$, we have that $\nu(\beta) = T$ as desired. Hence $\Sigma \vdash \beta$. □
Corollary
If $F\alpha$ and $\alpha \vdash = \beta$, then $F\beta$.

Proof. Assume $F\alpha$ and $\alpha \vdash = \beta$.
Then $\alpha \vdash = \beta$.
By the previous theorem (with $\Sigma = \emptyset$), we have $\vdash = \beta$.
\[ \square \]

(Duality of $\land$ and $\lor$)

Theorem (Exercise #9 on p 27)
Given a wff $\alpha$, using only the connectives $\neg, \lor, \land$, then define $\alpha^*$ recursively as:
- If $\alpha$ is a sentence symbol, then $\alpha^* = (\neg \alpha)$
- If $\alpha = (\neg \beta)$ then $\alpha^* = (\neg \beta^*)$
- If $\alpha = (\beta \land \gamma)$ then $\alpha^* = (\beta^* \land \gamma^*)$
- If $\alpha = (\beta \lor \gamma)$ then $\alpha^* = (\beta^* \lor \gamma^*)$

Then $\vdash (\alpha) = \vdash 1 \alpha^*$.

Proof. To show $\vdash (\alpha) = \vdash 1 \alpha^*$, it is sufficient to show $\neg \neg (\alpha) = \neg \neg (\alpha^*)$ for all truth assignments $v: S \rightarrow \{T, F\}$, where $S$ is the set of sentence symbols in $\alpha$. We do this by induction on $\alpha$.

Base case ($\alpha$ is sentence symbol):
$\alpha^*$ is $(\neg \alpha)$ so $\neg \neg (\alpha^*) = (\neg (\neg \alpha)) = (\neg \alpha)$

Induction step, $\alpha = (\neg \beta)$
There are two cases: $\neg \neg (\beta) = T$ and $\neg \neg (\beta) = F$
We can handle these with a truth table:

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\neg \beta^*$</th>
<th>$\neg (\neg \beta)$</th>
<th>$\neg (\neg \beta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
<td>$T$</td>
<td>$F$</td>
</tr>
</tbody>
</table>

Induction hyp
Induction step \( \alpha = (\beta \Box \gamma) \) where \( \Box \) is \( \land \) or \( \lor \).

By induction hypothesis \( \Box (\beta^n) = \Box (\neg \beta) \) and \( \Box (\gamma^n) = \Box (\neg \gamma) \).

\[
\begin{array}{c|c|c|c|c|c}
\beta \land \gamma & \neg (\beta \lor \gamma) & \beta^* \land \gamma^* & \neg (\beta \land \gamma) & \beta^* \lor \gamma^* \\
T \land T & F & T & T & F \\
T \land F & F & F & F & F \\
F \land T & F & T & T & F \\
F \land F & F & F & F & F \\
\end{array}
\]

By using a truth table, we've verified that

\( \Box (\alpha^n) = \Box (\neg \alpha) \).

\( \square \)

Remark: There are other variations where we move all the \( \neg \) next to a variable. For example, we can set

\( \neg (\neg \beta)^* = \beta^{**} \).

Compactness Theorem (p. 24 in book)

Let \( \Sigma \) be an infinite set of wffs such that for any finite subset \( \Sigma_0 \subseteq \Sigma \), there is a truth assignment that satisfies every member of \( \Sigma_0 \). Then there is a truth assignment which satisfies every member of \( \Sigma \).

Proof will come later...