Truth Assignments

We want to say what it means for a wff to be "always true" or "always false" or "sometimes true",

For example, \((A \land B)\) may be true or false (under the intended interpretation where \(\land\) means "and") depending on if \(A\) or \(B\) are true or false.

However, \((A \lor \neg A)\) is "always true" and \((A \land \neg A)\) is "always false".

We want to make this formal.

Definitions and notation

- \(F, T\) denote truth values (false, true)

- Let \(S\) be a set of sentence symbols

- A truth assignment is a function \(\nu: S \rightarrow \{F, T\}\). (For example, \(\nu(B) = T, \nu(A) = F\))

- Let \(\bar{S}\) denote the set of wffs that can be built up from \(S\) by the \(S\) formula building operations.
  (This is the same as the set of wffs containing only sentence symbols from \(S\).
  See HW problem #4 in Chapter 1.1)
* Extend \( \nu \) to \( \overline{\nu} : \mathcal{S} \to \{T, F\} \)

such that \( \overline{\nu} \) satisfies this recursive definition:

1. \( \overline{\nu}(\neg \alpha) = \begin{cases} T & \text{if } \overline{\nu}(\alpha) = F \\ F & \text{o/w} \end{cases} \)
2. \( \overline{\nu}(\alpha \land \beta) = \begin{cases} T & \text{if } \overline{\nu}(\alpha) = T \text{ and } \overline{\nu}(\beta) = T \\ F & \text{o/w} \end{cases} \)
3. \( \overline{\nu}(\alpha \lor \beta) = \begin{cases} T & \text{if } \overline{\nu}(\alpha) = T \text{ or } \overline{\nu}(\beta) = T \\ F & \text{o/w} \end{cases} \)
4. \( \overline{\nu}(\alpha \rightarrow \beta) = \begin{cases} F & \text{if } \overline{\nu}(\alpha) = T \text{ and } \overline{\nu}(\beta) = F \\ T & \text{o/w} \end{cases} \)
5. \( \overline{\nu}(\alpha \leftrightarrow \beta) = \begin{cases} F & \text{if } \overline{\nu}(\alpha) = \overline{\nu}(\beta) \\ T & \text{o/w} \end{cases} \)

Remarks

- \( \mathcal{N} \) is "inclusive or".
  - If \( \overline{\nu}(A) = T \) and \( \overline{\nu}(B) = T \) then \( \overline{\nu}(A \lor B) = T \).
  - (Example: \( 3 \) is odd) \( \overline{\nu}(3 \text{ is prime}) \) is true.

- \( \rightarrow \) is "material implication". If \( \overline{\nu}(A) = F \) then \( \overline{\nu}(A \rightarrow B) = T \) regardless of \( \overline{\nu}(B) \).

Divergence theorem

Example from calculus:

\[
\sum_{n=1}^{\infty} \text{(an diverges)} \rightarrow \left( \sum_{n=1}^{\infty} \text{an diverges} \right)
\]

This is true. So, it is true for \( a_0 = \frac{1}{n} \) and \( a_0 = 2^n \).

Example: \( P \) prime numbers, \( Q \) odd number

\( P \in Q \cup \{2\} \). By def of \( \subseteq \), \( (n \in P) \rightarrow (n \in Q \cup \{2\}) \)

This is still true for \( n = 9 \) and \( n = 4 \).

Example: By def of \( \subseteq \), \( \emptyset \subseteq A \) for any set \( A \).
The conditions can be summarized by this truth table:

<table>
<thead>
<tr>
<th>α</th>
<th>β</th>
<th>(¬α)</th>
<th>(α ∧ β)</th>
<th>(α ∨ β)</th>
<th>(α → β)</th>
<th>(α ↔ β)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
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<tr>
<td>T</td>
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<td>T</td>
</tr>
</tbody>
</table>

Calculating truth assignments:

With a tree:
If \( \vartheta(A) = T, \vartheta(B) = T, \vartheta(C) = F \) then

\[
\alpha = ((B \rightarrow (A \rightarrow C)) \leftrightarrow ((B \land A) \rightarrow C))
\]

\[
\begin{array}{c}
T \\
F \\
T \\
F \\
F \\
T \\
\end{array}
\]

With a truth table:

\[
\begin{array}{ccc}
(B \rightarrow (A \rightarrow C)) & ((B \land A) \rightarrow C) \\
T & F & T \\
T & F & F \\
T & T & T \\
T & T & F \\
T & T & F \\
\end{array}
\]

So \( \vartheta(\alpha) = T \).
Theorem 12A.
For any truth assignment $\nu$ on a set $S$, there is a unique function $\overline{\nu} : S \rightarrow \{F, T\}$ meeting the aforementioned conditions 0-5.

(Proof may come later.)

Remark. Why do we even need this last theorem? Imagine that we didn’t use parentheses. Then there are two ways to build $A \rightarrow B \rightarrow A$ and if $\nu(A) = F$, $\nu(A) = T$ then we have two values of $\overline{\nu}(A \rightarrow B \rightarrow A)$:

\[
\begin{align*}
A &\rightarrow B \rightarrow A \\
A &\rightarrow B \rightarrow A
\end{align*}
\]

\[
\begin{align*}
A \rightarrow B \rightarrow A &\rightarrow \neg A \\
A \rightarrow B \rightarrow A &\rightarrow \neg A
\end{align*}
\]