Homework 6
Due date: Friday March 4, 2016

• Complete this assignment on your own paper.
• Follow the rules for homework given as a handout on the first day of class.
• Start early and don’t be afraid to ask the instructor questions.
• Turn in on time. (However, if it must be late, I prefer correct and complete late homework over incorrect and incomplete timely homework.)

1 Properties about $\sim$ and $\preceq$.

For these you may use the following three results from the first homework.

• A one-to-one (injective) function composed with a one-to-one function is one-to-one.
• An onto (surjective) function composed with an onto function is onto.
• If there is a one-to-one (injective) function from $A$ to $B$, there is an onto (surjective) function from $B$ to $A$.

Any other facts about onto functions and one-to-one functions must be proven (even if you have seen them in another class).

#1. Show that $\sim$ is an equivalence relation. *(Hint: First write out what this means in terms of one-to-one and onto functions.)*

#2. Show that if $A \preceq B$ and $B \preceq C$ then $A \preceq C$. *(Hint: This should be easy after using results from your first homework assignment.)*

#3. Assume $A$ is nonempty. Show that $A \preceq B$ iff there exists an onto function $f : B \to A$. *(Hint: One direction you did on the first homework.)*
2 Countable sets

Recall that a set $A$ is countable if $A \leq \mathbb{N}$. If a set $A$ is infinite, there are three ways to show that $A$ is countable. (These three ways come from results we proved in class and from problem #3, above.)

- Give a one-to-one function $f : A \rightarrow \mathbb{N}$.
- Give an onto function $g : \mathbb{N} \rightarrow A$. (This is one of the easiest methods.)
- (If $A$ is infinite), give a method to enumerate all the elements on $A$. That is, give a way to list all the elements of $A$ as $a_0, a_1, a_2, \ldots$ (This is the same as giving a one-to-one and onto function $h : \mathbb{N} \rightarrow A$.)

Also by results from class, you can replace $\mathbb{N}$ with any countably infinite set (or example $\mathbb{Z}$). So, for example, you can show $A$ is countable by finding an onto function $h : \mathbb{N} \rightarrow A$.

#4. Show that $\mathbb{N} \times \mathbb{N}$ and $\mathbb{Z} \times \mathbb{Z}$ are countable. (Hint: Think of how to enumerate $\mathbb{Z} \times \mathbb{Z}$. You do not have to give me a formula. Just a general method—algorithm—to step through $\mathbb{Z} \times \mathbb{Z}$ touching each pair $(m,n)$ at least once.)

#5. Show that $\mathbb{Q}$ is countable. (Hint: One way to do this is to use #4 and then show that there is an onto function $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Q}$.)

#6. Show that a countable union of countable sets is countable. Specifically, let $(A_n)_{n \in \mathbb{N}}$ be a sequence of countable sets. Show that $\bigcup_n A_n$ is countable. (Hint: For each $n$, there is an onto function $f_n : \mathbb{N} \rightarrow A_n$. Then find an onto function $f : \mathbb{N} \times \mathbb{N} \rightarrow \bigcup_n A_n$.)

3 Sets of cardinality continuum

#7. Show that $[0,1] \sim [-1,1]$. (Hint: This is easy.)

#8. Show that $[-1,1] \sim \mathbb{R}$. (Hint: The easier method is to show that $[-1,1] \leq \mathbb{R}$ and $\mathbb{R} \leq [-1,1]$ and then use Cantor-Schröder-Bernstein. To show $\mathbb{R} \leq [-1,1]$, consider the function $f : \mathbb{R} \rightarrow [-1,1]$ given by $f(x) = \frac{x}{|x|+1}$, or alternately $f(x) = \frac{2}{\pi}\arctan(x).$)

#9. Show that $[0,1] \sim \{0,1\}^\infty$. (Hint: The easier method is to show that $[0,1] \leq \{0,1\}^\infty$ and $\{0,1\}^\infty \leq [0,1]$ and then use Cantor-Schröder-Bernstein. To show $[0,1] \leq \{0,1\}^\infty$ you can use that every real number in $[0,1]$ has at least one binary expansion. For example, $1/3 = .010101...$ and $1 = .11111...$. However, be careful, since some numbers like 1/2 have two binary expansions. To show, $\{0,1\}^\infty \leq [0,1]$, you can use the following function $f : \{0,1\}^\infty \rightarrow [0,1]$ where $f((b_0,b_1,b_2,\ldots)) = 0.b_0b_1b_20\ldots$)

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1In case you don’t know, every real number in the interval $[0,1]$ can be written out in binary. For example, $\pi/4 = .11001001000011...$. There are countably many special numbers (called dyadic rationals) where the binary expansion ends in all 0s, for example $3/8 = .011000000...$. In this case, if the last 1 is changed to a 0 and all the trailing 0s are changed to 1s, then it is the same number, for example $3/8 = .01011111111...$.

The only thing you need to know is that every real number has a binary expansion and if two different binary expansions correspond to the same number, $0.b_0b_1b_2... = 0.c_1c_2c_3...$, then one expansion ends in all 0s and the other ends in all 1s.