Homework 10
Due date: Wednesday April 13, 2016

- Complete this assignment on your own paper.
- Follow the rules for homework given as a handout on the first day of class.
- Start early and don’t be afraid to ask the instructor questions.
- Turn in on time. (However, if it must be late, I prefer correct and complete late homework over incorrect and incomplete timely homework.)

1 Definability and homomorphisms

#1. Consider the structure $\mathfrak{R} = (\mathbb{R}; +, \cdot)$. For each of the following sets, give a formula that defines that subset of $\mathfrak{R}$. (All such formulas should have one free variable. You can use all the symbols in $\exists, \forall, \land, \lor, \rightarrow, \neg, \iff, =$ as well as $+$ and $\cdot$. We don’t have $\leq$, so you need to use tricks from class to define it.)

(a) The interval $[0, \infty)$. ($\text{Hint:}$ We did something similar in class.)
(b) The set $\{1\}$. ($\text{Hint:}$ What special property does only 1 have?)
(c) The set $\{2\}$. ($\text{Hint:}$ Use that you can define 1.)
(d) The set $\{1/2\}$.
(e) The set $\{-\sqrt{2}, \sqrt{2}\}$. ($\text{Hint:}$ These are the set of solutions to the equation $x^2 + 2 = 0$.)
(f) The set $\{\sqrt{2}\}$.
(g) The interval $(0, 1)$.
(h) The interval $[\sqrt{2}, \infty)$.
(i) The union of intervals $(0, 1) \cup [\sqrt{2}, \infty)$.
(j) (Challenge) Every finite union of intervals of the form $(a, b)$, $[a, b]$, $(a, b]$, $[b, a]$, $(a, \infty)$, $[a, \infty)$, $(-\infty, b)$, $(-\infty, b]$ where $a$ and $b$ are algebraic numbers (that is, they are real number solutions to $p(x) = 0$ where $p(x)$ is a polynomial with integer coefficients). ($\text{Hint:}$ You do not have to give a formula but instead argue how you can construct such a formula. Using the ingredients above to show that you can define integers, rational numbers, the set of all solutions to polynomials with integer coefficients, algebraic numbers, intervals, and unions. It may be helpful to recall that for every polynomial $p(x)$, the equation $p(x) = 0$ has finitely many solutions, and therefore, between each pair of solutions is a rational number.)
Remark: Actually, every subset of $\mathbb{R}$ which is definable in $\mathbb{R}$ is of the form in part (j). In particular, this means that the only definable numbers in $\mathbb{R}$ are the algebraic numbers. This is an important result in model theory which we will not prove.

#2. State and prove the homomorphism theorem. Fill in the proof for terms, which we omitted in class.

#3. Consider the structure $\mathfrak{A} = (\mathbb{Z}; \leq, 0)$ and $\mathfrak{B} = (\mathbb{Z}; \geq, 0)$.

(a) Find a homomorphism $h$ of $\mathfrak{A}$ onto $\mathfrak{B}$. (Notice the “onto,” so $h$ must be an onto function.) Prove that $h$ is a homomorphism.

(b) Show there is only one homomorphism $h$ of $\mathfrak{A}$ onto $\mathfrak{B}$.\footnote{This is using the definition of homomorphism we gave in class and in the book which preserves both positive and negative information about predicates. Other books sometimes use “weak homomorphisms” which only preserve positive information. In that setting, this result would be false.}

(c) Is your map from part (a) an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$? Justify your answer.

#4. Consider the structure $(\mathbb{N}; \cdot)$.

(a) Recall that the fundamental theorem of arithmetic says that every positive natural number can be uniquely decomposed into prime numbers. For example, $84 = 2^23^15^07^1$. More generally, we can write a positive number $n$ as $n = p_1^{k_1}p_2^{k_2}p_3^{k_3}\ldots$ where $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ are the prime numbers listed in order, and $k_i$ is a natural number (where $k_i = 0$ for all but finitely many $i$). Consider the function $h: \mathbb{N} \rightarrow \mathbb{N}$ given by

$$h(n) = \begin{cases} 
0 & \text{if } n = 0 \\
p_2^{k_1}p_1^{k_2}p_3^{k_3} & \text{if } n = p_1^{k_1}p_2^{k_2}p_3^{k_3}\ldots
\end{cases}$$

Notice, this function decomposes $n$ into its prime components and then swaps all the 2s with 3s and all the 3s with 2s. Show that $h$ is an automorphism of $(\mathbb{N}; \cdot)$.

(b) Use the automorphism $h$ to show that the addition relation $\{\langle m, n, p \rangle \mid p = m + n\}$ is not definable in $(\mathbb{N}; \cdot)$.

2 Section 2.2 Problems (p. 99-104)

- Do problems 11, 16, and 28.