Handout: Provability and the soundness theorem

Much of this section is a repeat of the material for natural deduction on sentential logic. It is listed here for completeness, but we will go through it quickly.

1 Formal proofs via proof trees

Given a set \( \Sigma \) of wffs and a wff \( \varphi \), say that \( \varphi \) is provable from \( \Sigma \) (or \( \varphi \) is deducible from \( \Sigma \) or \( \varphi \) is a theorem of \( \Sigma \)), written \( \Sigma \vdash \varphi \), iff there is a proof tree proving \( \varphi \) from hypotheses (the uncovered leaves of the proof tree) in \( \Sigma \).

Remark 1. It is ok if there are elements of \( \Sigma \) not used as hypotheses in the proof tree for \( \Sigma \vdash \varphi \).

Remark 2. Just as with \( \models \), we have the following conventions.

- We write \( \vdash \varphi \) in place of \( \emptyset \vdash \varphi \).
- We write \( \psi, \theta \vdash \varphi \) in place of \( \{\varphi, \theta\} \vdash \varphi \).
- We write \( \Sigma; \theta \vdash \varphi \) or \( \Sigma, \theta \vdash \varphi \) to mean \( \Sigma \cup \{\theta\} \vdash \varphi \).

2 Inductive definition of \( \Sigma \vdash \varphi \).

However, to clearly prove results about the provability relation \( \vdash \), it is necessary to give an inductive definition of \( \Sigma \vdash \varphi \). This involves writing the same rules as before in a different format. We only need to list the original rules, not the derived one.

Definition 3. The provable hypotheses-theorem pairs are those pairs \( \Sigma \vdash \varphi \) (where \( \Sigma \) is finite) is inductively defined by the following rules.

- Base rule: (This rule says \( \Sigma; \varphi \vdash \varphi \) is provable. Equivalently, it says if \( \varphi \in \Sigma \) then \( \Sigma \vdash \varphi \) is provable.)
  \[
  \frac{}{\Sigma; \varphi \vdash \varphi} \quad \text{assumption}
  \]

- Induction rules: (These rules say that if the hypotheses-theorem pairs above the line are provable, then the hypotheses-theorem pair below the line is provable.)
  \[
  \frac{\Sigma; \varphi \vdash \psi}{\Sigma \vdash \varphi \rightarrow \psi} \quad \rightarrow I \quad \frac{\Sigma \vdash \varphi \rightarrow \psi \quad \Sigma \vdash \varphi}{\Sigma \vdash \varphi} \quad \rightarrow E
  \]
  \[
  \frac{\Sigma; \neg \varphi \vdash \bot}{\Sigma \vdash \varphi} \quad \text{RAA}
  \]

1
The quantifier rules come with the same side conditions as before, namely for \( \forall I \):

- \( x \) is not free in any formula in \( \Sigma \),
- \( y \) is substitutable for \( x \) in \( \varphi \), and
- either \( y \) is not free in \( \varphi \), or \( y = x \).

For \( \forall E \):

- \( t \) is substitutable for \( x \) in \( \varphi \).

**Definition 4.** A proof tree is formally defined inductively as follows:

- **Base rule:** The following is a proof tree: \( \Sigma; \varphi \vdash \varphi \)
- **Induction rules:**
  - \((\rightarrow E)\) If \( \Sigma \vdash \varphi \) and \( \Sigma \vdash \psi \) are proof trees, then the following is a proof tree:
    \[
    \frac{\vdots}{\Sigma \vdash \varphi \rightarrow \psi} \quad \frac{\vdots}{\Sigma \vdash \varphi} \quad \rightarrow E
    \]
  - Similarly for the other rules in Definition 3. (The quantifier rules have to obey the side conditions.)

**Remark 5.** In these last two definitions, we assumed that \( \Sigma \) is finite. This was done for technical reasons. We want to assume proof tree are finite objects and that they use finitely many variables.

**Definition 6.** If \( \Sigma \) is infinite, then say we say that \( \Sigma \) proves \( \varphi \) (written \( \Sigma \vdash \varphi \)) iff \( \Sigma' \vdash \varphi \) for some finite subset \( \Sigma' \subseteq \Sigma \).
3 Theorems about proofs

Just as in sentential logic, we can do induction on proofs. It is the same basic idea as in
sentential logic, except that we may need to use the following trick when working with ∀I.
Suppose we have a proof of \( \Sigma \vdash \forall y \varphi(y) \) ending in

\[
\frac{\Sigma \vdash \varphi(x)}{\Sigma \vdash \forall y \varphi(y)} \text{ ∀I}
\]

(Here we write \( \varphi(x) \) in place of \( \varphi \) to emphasize the free variable \( x \) and we write \( \varphi(y) \) in place of \( \varphi_x \) to emphasize that we just replaced all free \( x \) with \( y \).) Now, the choice of variable \( x \) doesn’t matter as long as the variable we use obeys all the side conditions—in particular, \( x \) must not be free in \( \Sigma \). If we replace \( x \) with a “fresh” variable \( z \) occurring nowhere else in the
proof, we still have a proof of \( \Sigma \vdash \forall y \varphi(y) \)

\[
\frac{\Sigma \vdash \varphi(z)}{\Sigma \vdash \forall y \varphi(y)} \text{ ∀I}
\]

where we replaced all free \( x \) with \( z \) in the original proof. When do the ∀I step in induction
on proofs, it is OK (and useful) to assume \( x \) is sufficiently “fresh.” For example, let us look
at the proof of weakening.

**Proposition 7 (Weakening).** If \( \Sigma \vdash \varphi \) and \( \Sigma \subseteq \Delta \) (both finite), then \( \Delta \vdash \varphi \).

**Proof.** The proof is the same as in the sentential logic. The only case that is special is the
rule ∀I.

- Assume we have a proof of \( \Sigma \vdash \forall y \varphi(y) \) ending in

\[
\frac{\Sigma \vdash \varphi(x)}{\Sigma \vdash \forall y \varphi(y)} \text{ ∀I}
\]

Since we may assume \( x \) is sufficiently fresh, assume that \( x \) is also not in \( \Delta \). Then by
the induction hypothesis we have a proof of \( \Delta \vdash \varphi(x) \). Now, the following gives us a
proof of \( \Delta \vdash \forall y \varphi(y) \).

\[
\frac{\Delta \vdash \varphi(x)}{\Delta \vdash \forall y \varphi(y)} \text{ ∀I}.
\]

These next results are similar to those proved in sentential logic.

**Proposition 8 (Deduction Theorem).** \( \Sigma; \varphi \vdash \psi \) iff \( \Sigma \vdash \varphi \rightarrow \psi \).

**Proof.** This is the same proof as in sentential logic. \( \square \)

**Proposition 9 (Finite character).** If \( \Sigma \vdash \varphi \) then there is a finite set \( \Sigma' \subseteq \Sigma \) such that \( \Sigma' \vdash \varphi \).

**Proof.** If \( \Sigma \) is finite, then let \( \Sigma' = \Sigma \). If \( \Sigma \) is infinite, then this is just the definition of
\( \Sigma \vdash \varphi \).

We will prove more such results as we need them.
4 Consistency and satisfiability

Just as in sentential logic, we have the following definitions.

**Definition 10.** A set of wffs $\Sigma$ is **consistent** iff $\Sigma \not\vdash \bot$ (that is $\bot$ is not provable from $\Sigma$).

**Definition 11.** A set of wffs $\Sigma$ is **satisfiable** iff there is a structure $\mathfrak{A}$ and a function $s : V \to |\mathfrak{A}|$ such that for all $\varphi \in \Sigma$ we have $|=\mathfrak{A} \varphi[s]$. The set $\Sigma$ is **finitely satisfiable** iff every finite subset $\Sigma' \subseteq \Sigma$ is satisfiable.

Just as in sentential logic, we have following three important theorems.

**Soundness Theorem.** For a set $\Sigma$ of wffs and a wff $\varphi$, we have the following.
1. If $\Sigma \vdash \varphi$, then $\Sigma \models \varphi$.
2. If $\Sigma$ is satisfiable, then $\Sigma$ is consistent.

**Gödel's Completeness Theorem.** For a set $\Sigma$ of wffs and a wff $\varphi$, we have the following.
1. If $\Sigma \models \varphi$, then $\Sigma \vdash \varphi$.
2. If $\Sigma$ is consistent, then $\Sigma$ is satisfiable.

**Compactness Theorem.** For a set $\Sigma$ of wffs and a wff $\varphi$, we have the following.
1. If $\Sigma \models \varphi$ then there is a finite set $\Sigma' \subseteq \Sigma$ such that $\Sigma' \vdash \varphi$.
2. If $\Sigma$ is finitely satisfiable then $\Sigma$ is satisfiable.

We will give the proofs of the Soundness and Completeness Theorems soon. The the Compactness Theorem follows from the Soundness and Completeness Theorems. We leave this result and other basic results as exercises. (All of these have proofs similar to those in sentential logic.)

**Exercise 12.** Prove the following.
1. $\Sigma$ is satisfiable iff $\Sigma \not\models \bot$.
2. $\Sigma \models \varphi$ iff $\Sigma \cup \{\neg \varphi\}$ is unsatisfiable.
3. $\Sigma \models \varphi$ iff $\Sigma \cup \{\neg \varphi\}$ is inconsistent.
4. Versions (1) and (2) of Soundness Theorem are provable from each other (without proving the Soundness Theorem first).
5. Versions (1) and (2) of Completeness Theorem are provable from each other (without proving the Completeness Theorem first).
6. Versions (1) and (2) of Compactness Theorem are provable from each other (without proving the Compactness Theorem first).
7. Version (1) of the Compactness Theorem is provable from versions (1) of the Soundness and Completeness Theorems. (Hint: Use the finite character of ⊢.)

Remark 13. The Compactness Theorem is one of the most important theorems in all of logic. It has many applications—not only in logic, but also in other areas of mathematics. We will touch on these applications slightly, but we will not have time to appreciate the true power of this theorem.

Also, like sentential logic, there is a direct proof of the compactness theorem that does not go through the Soundness and Completeness Theorems. It requires Zorn’s Lemma (a version of the Axiom of Choice) and a very powerful (some might say “magical”) mathematical structure called an ultrafilter. (If you are interested in the details, ask me.)

5 Proof of the Soundness theorem

To prove the soundness theorem we need two lemmas.

Lemma 14 (Substitution lemma). Let \( \mathfrak{A} \) be a structure and \( s: V \rightarrow |\mathfrak{A}| \).

1. For a terms \( u \) and \( t \) and a variable \( x \),
   \[
   \bar{s}(u^x_t) = \bar{s}(x | \bar{s}(t))(u).
   \]

2. If the term \( t \) is substitutable for \( x \) in \( \varphi \) then
   \[
   \models_\mathfrak{A} \varphi^x_t[s] \iff \models_\mathfrak{A} \varphi[s(x | \bar{s}(t))].
   \]

Proof. (1) Do this by induction on the term \( u \):

- If \( u \) is a variable \( y \): Then
  \[
  \bar{s}(u^x_t) = \bar{s}((y)^x_t) = \begin{cases} \bar{s}(t) & \text{if } y = x \\ \bar{s}(y) & \text{if } y \neq x \end{cases}
  \]
  and
  \[
  \bar{s}(x | \bar{s}(t))(u) = \bar{s}(x | \bar{s}(t))(y) = \begin{cases} \bar{s}(t) & \text{if } y = x \\ \bar{s}(y) & \text{if } y \neq x \end{cases}
  \]

So \( \bar{s}(u^x_t) = \bar{s}(x | \bar{s}(t))(u) \).

- If \( u \) is \( f u_1 \ldots u_n \): Then
  \[
  \bar{s}(u^x_t) = \bar{s}(f(u_1)^x_t \ldots (u_n)^x_t) = f^\mathfrak{A}(\bar{s}((u_1)^x_t), \ldots, \bar{s}((u_n)^x_t)) = f^\mathfrak{A}(\bar{s}(x | \bar{s}(t))(u_1), \ldots, \bar{s}(x | \bar{s}(t))(u_n)) = \bar{s}(x | \bar{s}(t))(f u_1 \ldots u_n) = \bar{s}(x | \bar{s}(t))(u)
  \]

5
(2) Do this by induction on ϕ:

- If ϕ is Pu₁…un: Then
  \[\models_\mathfrak{A} \varphi^x_{i}[s] \iff \models_\mathfrak{A} P(u_1)^x_1 \ldots (u_n)^x_n [s] \]
  \[\iff \langle s(u_1)^x_1, \ldots, (u_n)^x_n \rangle \in P^\mathfrak{A} \]
  \[\iff \langle s(x | \bar{s}(t))(u_1), \ldots, s(x | \bar{s}(t))(u_n) \rangle \in P^\mathfrak{A} \quad \text{IH} \]
  \[\iff \models_\mathfrak{A} P u_1 \ldots u_n [s(x | \bar{s}(t))] \]
  \[\iff \models_\mathfrak{A} \varphi [s(x | \bar{s}(t))] \]

- If ϕ is α → β: Then
  \[\models_\mathfrak{A} \varphi^x_{i}[s] \iff \models_\mathfrak{A} (\alpha^x_1 \rightarrow \beta^x_1) [s] \]
  \[\iff (\models_\mathfrak{A} \alpha^x_1 [s] \text{ implies } \models_\mathfrak{A} \beta^x_1 [s]) \]
  \[\iff (\models_\mathfrak{A} \alpha [s(x | \bar{s}(t))] \text{ implies } \models_\mathfrak{A} \beta [s(x | \bar{s}(t))]) \quad \text{IH} \]
  \[\iff \models_\mathfrak{A} (\alpha \rightarrow \beta) [s(x | \bar{s}(t))] \]
  \[\iff \models_\mathfrak{A} \varphi [s(x | \bar{s}(t))] \]

- If ϕ is ∀x α: Then
  \[\models_\mathfrak{A} \varphi^x_{i}[s] \iff \models_\mathfrak{A} \forall x \alpha [s] \]
  \[\iff \models_\mathfrak{A} \forall x \alpha [s(x | \bar{s}(t))] \quad \text{only depends on free variables} \]
  \[\iff \models_\mathfrak{A} \varphi [s(x | \bar{s}(t))] \]

- If ϕ is ∀y α (y ≠ x): Then
  \[\models_\mathfrak{A} \varphi^x_{i}[s] \iff \models_\mathfrak{A} \forall y \alpha^x_{i}[s] \]
  \[\iff \text{for all } a \in |\mathfrak{A}|, \models_\mathfrak{A} \alpha^x_{i} [s(y | a)] \]
  \[\iff \text{for all } a \in |\mathfrak{A}|, \models_\mathfrak{A} \alpha^x_{i} [s(y | a)(x | \bar{s}(t))] \quad \text{IH} \]
  \[\iff \models_\mathfrak{A} \forall y \alpha^x_{i} [s(x | \bar{s}(t))] \]
  \[\iff \models_\mathfrak{A} \varphi [s(x | \bar{s}(t))] \]

\[\square\]

Lemma 15. Every inference rule holds with \(\models\) replaced by \(\models_\mathfrak{A}\).

Proof. This is easy to do for most rules by just using the definition of satisfiable and \(\models_\mathfrak{A}\). We will just do the quantifier rules and leave the rest as exercises. \[\square\]

- ∀I: Assume the side conditions that x does not occur in Σ, y does not occur free in ϕ (unless \(y = x\)), and y is substitutable for x in ϕ. We need to show that \(\Sigma \models \varphi\) implies \(\Sigma \models \forall y \varphi^x_y\).
  Assume \(\Sigma \models \varphi\). We want to show that \(\Sigma \models \forall y \varphi^x_y\). Consider an arbitrary structure \(\mathfrak{A}\) and an arbitrary function \(s: V \rightarrow |\mathfrak{A}|\). Assume that \(\models_\mathfrak{A} \psi [s]\) for all \(\psi\) in \(\Sigma\). We
want to show that $\models_{\mathfrak{A}} \forall y \varphi^r_y [s]$. Therefore we need to show that $\models_{\mathfrak{A}} \varphi^r_y [s(y \mid a)]$ for all $a \in |\mathfrak{A}|$. Fix an arbitrary $a$. Since $\models_{\mathfrak{A}} \psi [s]$ for all $\psi$ in $\Sigma$ and since $x$ is not free in $\Sigma$, we have that $\models_{\mathfrak{A}} \psi [s(x \mid a)]$ for all $\psi$ in $\Sigma$. Since $\Sigma \models \varphi$, we have that $\models_{\mathfrak{A}} \varphi [s(x \mid a)]$. Now let’s handle two cases:

- If $y = x$, then we are done since we showed that $\models_{\mathfrak{A}} \varphi^r_y [s(y \mid a)]$ (which is the same as $\models_{\mathfrak{A}} \varphi [s(x \mid a)]$) for an arbitrary $a$.

- If $y \neq x$, then since either $y$ is not free in $\varphi$, we have that $\models_{\mathfrak{A}} \varphi [s(y \mid a)(x \mid a)]$. By the Substitution Lemma, this is equivalent to $\models_{\mathfrak{A}} \varphi^r_y [s(y \mid a)]$ which is what we wanted to prove.

• $\forall E$: Assume $y$ is substitutable for $x$ in $\varphi$. We need to show that $\Sigma \models \forall x \varphi$ implies $\Sigma \models \varphi^r_x$.

Assume $\Sigma \models \forall x \varphi$. We want to show that $\Sigma \models \varphi^r_x$. Consider an arbitrary structure $\mathfrak{A}$ and an arbitrary function $s : V \to |\mathfrak{A}|$. Assume that $\models_{\mathfrak{A}} \psi [s]$ for all $\psi$ in $\Sigma$. We want to show that $\models_{\mathfrak{A}} \varphi^r_x [s]$. Since $\models_{\mathfrak{A}} \psi [s]$ for all $\psi$ in $\Sigma$, and since $\Sigma \models \forall x \varphi$, we have that $\models_{\mathfrak{A}} \forall x \varphi [s]$. Therefore, it follows that $\models_{\mathfrak{A}} \varphi [s(x \mid \bar{s}(t))]$. By the Substitution Lemma, this is equivalent to $\models_{\mathfrak{A}} \varphi^r_x [s]$ which is what we wanted to prove.

The Soundness Theorem follows from this previous lemma by induction on proofs. See the proof of the Soundness Theorem for sentential logic.