A uniform reducibility in computably presented Polish spaces

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www.personal.psu.edu/jmr71/

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Begin at the end: Computable arcs

Computable arcs in $\mathbb{R}^2$

Definition

- An **arc** in $\mathbb{R}^2$ is a path which doesn't self intersect.
- A **computable arc** in $\mathbb{R}^2$ is the image of an injective computable function $f\colon [0,1] \to \mathbb{R}^2$.

Question (McNicholl)

Which pairs are members of some computable arc?

Our approach

To answer this, we will use **a new reducibility**, motivated in **three ways**.

Motivation One
Extending truth table reducibility to Polish spaces

Turing reducibility and truth table reducibility

Turing reduction and Turing reducibility ($\leqslant_T$)

- A **Turing reduction** is a partial computable map $\Phi\colon \subseteq 2^{\mathbb{N}} \to 2^{\mathbb{N}}$.
- $X \leqslant_T Y$ ($X, Y \in 2^{\mathbb{N}}$) if $\Phi(Y) = X$ for some Turing reduction Φ.

Truth table reduction and truth table reducibility ($\leqslant_{tt}$)

- A **truth-table reduction** is a total computable map $\Phi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$.
- $X \leqslant_{tt} Y$ ($X, Y \in 2^{\mathbb{N}}$) if $\Phi(Y) = X$ for some truth-table reduction Φ.

- Many books define truth table reducibility via
  - computable bounds on the number of steps in the computation, or
  - truth-tables.
- The equivalence of these definitions requires the compactness of $2^{\mathbb{N}}$.

What are the "reals"?

The "reals"

- Cantor space $2^{\mathbb{N}}$ — computability theoretic "reals"
- Baire space $\mathbb{N}^{\mathbb{N}}$ — set theoretic "reals"
- Euclidean space $\mathbb{R}$ — real? "reals"
- A Polish space $\mathbb{X}$ — descriptive set theoretic "reals"

Computability theory on $\mathbb{R}$ (and Polish spaces)

- Every real $x \in \mathbb{R}$ can be encoded by a **Cauchy name**: a sequence of rationals $\{q_n\}$ (coded as a function $f \in \mathbb{N}^{\mathbb{N}}$) such that $|x - q_n| \leqslant 2^{-n}$.
- $x \in \mathbb{R}$ **computes** $y \in \mathbb{R}$ if there is a Turing reduction $\Phi\colon \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ which maps every Cauchy name for x to a Cauchy name for y.
- This can be extended to every **computably presented Polish space**, a.k.a. a **computable Polish space**, a **computable metric space**.

Turing degrees on the "reals"

- $\mathbb{N}^{\mathbb{N}}$ under Turing reduction
  - Same Turing degrees as $2^{\mathbb{N}}$
  - Identify each function $f \in \mathbb{N}^{\mathbb{N}}$ with its graph
- $\mathbb{R}$ under computability
  - Same Turing degrees
  - Identify each real $r \in \mathbb{R}$ with its binary expansion
- Computable Polish space (e.g. $[0,1]^{\mathbb{N}}$ or $C(2^{\mathbb{N}})$)
  - **NOT** the same Turing degrees! (J. Miller)
  - One gets a larger degree structure called the **continuous degrees**.

Question

What about truth table reducibility on these spaces?

Extending truth-table reducibility to $\mathbb{N}^{\mathbb{N}}$

Problem

Which definition of truth table reducibility do we use on $\mathbb{N}^{\mathbb{N}}$?

Solution

The essence of truth table reduction is its **uniform nature**.

Uniform reducibility on $\mathbb{N}^{\mathbb{N}}$

- $f \in \mathbb{N}^{\mathbb{N}}$ is **uniformly reducible**[1] to $g \in \mathbb{N}^{\mathbb{N}}$ (written $f \leqslant_{unif} g$) if there is a total computable map $\Phi\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\Phi(g) = f$.
- Denote the corresponding degree structure as $\mathbf{deg}_{unif}(\mathbb{N}^{\mathbb{N}})$.

Truth-table reducibility vs uniform reducibility

- On $2^{\mathbb{N}}$, uniform reducibility is just truth-table reducibility.

[1] Tentative terminology. Medvedev reducibility is sometimes called "uniform reducibility."

The new degrees in $\mathbb{N}^{\mathbb{N}}$

Proposition

- $\mathbf{deg}_{unif}(2^{\mathbb{N}})$ (truth-table degrees) is a proper subideal of $\mathbf{deg}_{unif}(\mathbb{N}^{\mathbb{N}})$.
- Every $f \in \mathbb{N}^{\mathbb{N}}$ uniformly computes $\mathrm{graph}(f) \in 2^{\mathbb{N}}$.
- The following are equivalent:
  - f has truth-table degree ($\mathbf{deg}_{unif}(f) \in \mathbf{deg}_{unif}(2^{\mathbb{N}})$).
  - $f \equiv_{unif} \mathrm{graph}(f)$.
  - f is dominated by a computable function.

Extending truth-table reducibility to $\mathbb{R}$

Uniform reducibility on $\mathbb{R}$

- $x \in \mathbb{R}$ is **uniformly reducible** to $y \in \mathbb{R}$ (written $x \leqslant_{unif} y$) if there is a total computable map $\Phi\colon \mathbb{R} \to \mathbb{R}$ such that $\Phi(y) = x$.
- Denote the corresponding degree structure as $\mathbf{deg}_{unif}(\mathbb{R})$.

Question

How does $\mathbf{deg}_{unif}(\mathbb{R})$ compare to $\mathbf{deg}_{unif}(2^{\mathbb{N}})$

To do

We need a way to compare elements between difference spaces...

Computable analysis for computable Polish spaces

- Σ^0_1 **sets** (**effectively open sets**) are computable unions of basic open balls.
- Π^0_1 **sets** (**effectively closed sets**) are complements of Σ^0_1 sets.
- A point is a **weak 1-generic** if it is found in every dense Σ^0_1 set.

Uniform reducibility between Polish spaces

Problem

All total computable maps of type $\Phi\colon \mathbb{R} \to 2^{\mathbb{N}}$ are **constant**!

Solution

Use **partial** computable maps with **with a Π^0_1 domain**.

Uniform reducibility for computable Polish spaces $\mathbb{X}$ and $\mathbb{Y}$

- $x \in \mathbb{X}$ is **uniformly reducible** to $y \in \mathbb{Y}$ (written $x \leqslant_{unif} y$) if there is a partial computable map $\Phi\colon \subseteq \mathbb{Y} \to \mathbb{X}$ with a Π^0_1 domain such that $\Phi(y) = x$.
- $\mathbf{deg}_{unif} := \bigcup \mathbf{deg}_{unif}(\mathbb{X})$ for all computable Polish spaces $\mathbb{X}$.

Proposition

Every partial computable map $\Phi\colon \subseteq 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ with a Π^0_1 domain can be extended to a total computable map $\overline{\Phi}\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$. Same for $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{R}$.

Computability vs. uniform computability

Equivalent definitions of **computable**

- y **computes** x
- Exists **partial** comp. Φ which maps every name for y to a name for x.
- Exists partial comp. $\Phi\colon \subseteq \mathbb{Y} \to \mathbb{X}$ with Π^0_2 **domain** such that $\Phi(y) = x$.
- Every name for y **computes** a name for x.
- Exists algorithm which takes in a (name for) an input and either
  - enumerates (a name for) the output, or
  - **does not halt**

Equivalent definitions of **uniformly computable**

- y **uniformly computes** x ($x \leqslant_{unif} y$)
- Exists **total** comp. Φ which maps every name for y to a name for x.
- Exists partial comp. $\Phi\colon \subseteq \mathbb{Y} \to \mathbb{X}$ with Π^0_1 **domain** such that $\Phi(y) = x$.
- Every name for y **uniformly computes** a name for x.
- Exists algorithm which takes in a (name for) an input and either
  - enumerates (a name for) the output, or
  - **halts with an error**

Comparing $\mathbf{deg}_{unif}(2^{\mathbb{N}})$ and $\mathbf{deg}_{unif}(\mathbb{R})$

Proposition

- $\mathbf{deg}_{unif}(2^{\mathbb{N}}) \subseteq \mathbf{deg}_{unif}(\mathbb{R})$.
- Every $X \in 2^{\mathbb{N}}$ uniformly computes $0.X \in \mathbb{R}$.
- The following are equivalent for $x \in \mathbb{R}$
  - x has truth-table degree
  - $x \equiv_{unif}$ binary-expansion(x)
  - x is not weak-1-generic in $\mathbb{R}$

Many, many questions

- What is the structure of $\mathbf{deg}_{unif}(\mathbb{X})$ for various spaces $\mathbb{X}$?
- What is the $\mathbf{deg}_{unif}$-structure of various classes of points?
  - completions of PA and diagonally noncomputable functions
  - left c.e. (lower semicomputable) reals, limit computable points
  - randoms and generics
  - K-trivials and other lowness classes
  - etc.

Motivation Two
Schnorr randomness and weak 1-genericity

Randomness and Genericity

	Relativization	Uniform Relativization
Martin-Löf random	MLR^A	
1-generic	$1\mathrm{G}^A$	
Schnorr random	SR^A	uniform-SR^A
Weak 1-generic	$\mathrm{W1G}^A$	uniform-$\mathrm{W1G}^A$

Theorem (Miyabe, Miyabe-R.)

The following are equivalent for $A, B \in 2^{\mathbb{N}}$.

- $A \oplus B$ is Schnorr random
- A is Schnorr random and B is Schnorr random uniformly relative to A

Uniform relativization and oracles

Oracles as information

For oracles $a \in \mathbb{X}$ and $b \in \mathbb{Y}$,

- If a **computes** b and X is $\mathbf{MLR}^a$, then X is $\mathbf{MLR}^b$.
- If a **computes** b and X is $\mathbf{1G}^a$, then X is $\mathbf{1G}^b$.
- If a **unif. comp.** b and X is $\mathbf{uniform\text{-}SR}^a$, then X is $\mathbf{uniform\text{-}SR}^b$.
- If a **unif. comp.** b and X is $\mathbf{uniform\text{-}W1G}^a$, then X is $\mathbf{uniform\text{-}W1G}^b$.

Schnorr randomness in the uniform degrees

Definition

$$\mathsf{Schnorr} := \left\{ (x, \mathbb{X}) \,\middle|\, x \in \mathbb{X} \text{ is } \mu\text{-Schnorr random, } \mu \text{ computable measure} \right\}$$

Theorem (McNicholl-R. building on Bienvenu-Porter)

- $\mathsf{Schnorr}$ is a closed $\leqslant_{unif}$-downward.
- $\mathbf{deg}_{unif}(\mathsf{Schnorr}) \subseteq \mathbf{deg}_{unif}(2^{\mathbb{N}})$

Motivation Three
Points in computable arcs

Computable arcs in $\mathbb{R}^2$

Definition

- An **arc** in $\mathbb{R}^2$ is a path which doesn't self intersect
- A **computable arc** in $\mathbb{R}^2$ is the image of an injective computable function $f\colon [0,1] \to \mathbb{R}^2$.

Question (McNicholl)

Which pairs are members of some computable arc?

Members of computable arcs have $\mathbb{R}$ uniform degree

Theorem (McNicholl-R.)

The following are equivalent for a vector $x \in \mathbb{R}^2$:

- x a member of a computable arc
- x has the same uniform degree as some $r \in \mathbb{R}$

Proof.

Proof by picture. See board. □

Which pairs are found in some computable arc?

Theorem (McNicholl-R.)

- No weak 1-generic in $\mathbb{R}^2$ is in a computable arc.
  - It is well known that arcs are have dense complements.
- A pair is in a computable arc iff it has $\mathbb{R}$ uniform degree.
- Every pair of truth-table degree is in a computable arc.
- Every μ-Schnorr random (μ computable measure on $\mathbb{R}^2$) is in a computable arc.

Conclusion

In summary

This new reducibility ...

- ... is a natural extension of truth-table reducibility which parallels Turing reducibility and its generization to computable Polish spaces
- ... is a useful tool for working with concepts like Schnorr randomness and weak 1-genericity
- ... allows us to connect and characterize new sets of points such as members of computable arcs

Thank You!

These slides will be available on my webpage:

http://www.personal.psu.edu/jmr71/

Or just Google™ me, “Jason Rute”.

P.S. I am on the job market.