An application of computable continuous model theory to a question in proof theory

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www.personal.psu.edu/jmr71/

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Convergence

Convergence theorems

Monotone convergence principle

Let (X, d) be a complete metric space with a linear order $<$ satisfying

$$x < y < z \quad \to \quad d(x,z) = d(x,y) + d(y,z).$$

Any bounded nondecreasing sequence $(c_n)_{n \in \mathbb{N}}$ converges.

Mean ergodic theorem

Let $(X, \|\cdot\|)$ is a reflexive Banach space with a nonexpansive linear transformation $T\colon X \to X$,

$$T(ax+y) = aT(x) + T(y) \quad \text{and} \quad \|T(x)\| \leqslant \|x\|.$$

Then for $c \in X$, the ergodic averages $\frac{1}{n}\sum_{k<n} T^k(c)$ converge.

Our setup

- Let $\mathcal{M}$ be a complete metric space (X, d) with possible additional structure.
- Let $(c_n)_{n\in\mathbb{N}}$ is a distinguished sequence in X.
- Let P be a property that could hold of $(\mathcal{M}, (c_n)_{n\in\mathbb{N}})$.

Convergence theorem template

If P holds of the pair $(\mathcal{M}, (c_n)_{n\in\mathbb{N}})$, then c_n converges.

Question

For which properties P is the rate of convergence

- **uniform**—exists single rate for all pairs $(\mathcal{M}, (c_n)_{n\in\mathbb{N}})$ satisfying P?
- **computable**—rate is computable uniformly from $(\mathcal{M}, (c_n)_{n\in\mathbb{N}})$?
- **computably uniform**—exists single computable uniform rate?

Bait-and-switch

- This talk is not about usual Cauchy rates of convergence...

Metastable convergence

Three ways to say converge

The following are all equivalent ways to say that $(c_n)_{n\in\mathbb{N}}$ converges.

- $(c_n)_{n\in\mathbb{N}}$ is Cauchy **(contains a lot of information, but not very uniform)**

$$\underbrace{\forall \varepsilon > 0\ \exists m \in \mathbb{N}}_{\text{rate of convergence}}\ \forall n, n' \geqslant m\ d(c_n, c_{n'}) < \varepsilon.$$

- $(c_n)_{n\in\mathbb{N}}$ has finitely many ε-jumps

$$\underbrace{\forall \varepsilon > 0\ \exists n \in \mathbb{N}}_{\text{rate of convergence}}\ \forall u_0 \leqslant v_0 \leqslant u_1 \leqslant v_1 \leqslant \ldots \leqslant v_{n-1} \leqslant v_{n-1}$$

$$\exists k \in [0, n-1]\ d(c_{u_k}, c_{v_k}) < \varepsilon.$$

  - (Similar to upcrossing bounds and variational bounds.)

- $(c_n)_{n\in\mathbb{N}}$ is metastable **(very uniform, but contains little information)**

$$\underbrace{\forall \varepsilon > 0\ \forall F : \mathbb{N} \to \mathbb{N}\ \exists m \in \mathbb{N}}_{\text{rate of convergence}}\ \forall n, n' \in [m, F(m)]\ d(c_n, c_{n'}) < \varepsilon.$$

Why metastability?

- Analysis
  - Any rate of convergence is better than no rate.
  - Rates of metastable convergence are more uniform.
  - Better rates may not be known (or even possible?):
  - May give simplest or most accessible proof of convergence.
  - An alternative to nonstandard analysis.
  - Example: Tao's ergodic theorem for multiple commuting averages.
- Logic
  - Metastable rates are computable.
  - Metastable convergence theorems are constructive.
  - Proof theoretic methods exist to extract metastable bounds: proof mining.
  - Closely connected to ultraproducts and nonstandard analysis.
  - **Uniform metastable rates can be computed from the statement of the theorem alone! (This talk.)**

Results: prior and new

Let P be a property that could hold of an arbitrary metric structure with a distinguished subsequence: $\mathbb{X} = (X, d, \ldots, \{c_n\}_{n\in\mathbb{N}})$. Consider a theorem:

($*$) If P holds of $\mathbb{X}$ then c_n converges.

- Kohlenbach. *Some logical metatheorems with applications to functional analysis.* Trans AMS, 2004.
  - If (1) the theorem ($*$) is provable in $A^\omega[X,d]$,[1] and
  - (2) P is expressible by a $\forall$-formula (basically a Π^0_1 property), then
  - there there exists a computable uniform metastable rate of convergence (uniformly extractable from the proof of ($*$) in $A^\omega[X,d]$).
- Avigad, Iovino. *Ultraproducts and metastability.* NYJM, 2013.
  - If $\mathcal{C} = \{\mathbb{X} : \mathbb{X} \text{ satisfies } P\}$ is closed under ultraproducts, then
  - there is a uniform rate of metastable convergence.
- R. (This talk)
  - If P is axiomitizable by a set of sentences Σ in continuous logic, then
  - there is a uniform rate of metastable convergence computable from Σ.

[1] $A^\omega[X,d]$ is a type theory extending $\mathsf{PA}+\mathsf{DC}$ with a type for X and axioms for the metric d.

Main result

Continuous logic

- Continuous logic is a logic for dealing with “metric structures.”
- There have been many variants over the years.
  - Chang and Keisler 1966
  - Henson, et al.
  - etc.
- The current variant is due to Ben Yaacov.
- Ben Yaacov’s version very much resembles first-order logic!
  - Compactness, completeness, Lowenheim-Skolem, ultraproducts, etc.
- See survey article *Model theory for metric structures* for a good introduction.

First-order logic vs. Continuous first-order logic

	First-Order Logic	Continuous First-Order Logic
Universe	set $(M,=)$	comp. bdd. metric space (M,d) $(d(x,y) \leqslant 1)$
Truth values	T and F	$[0,1]$ $(0 = \mathit{true}, 1 = \mathit{false})$
Func. symbol	symbol f (arity n)	symbol f (arity n and mod. of cont. $\delta(\varepsilon)$)
Functions	$f^{\mathcal{M}} : M^n \to M$	$f^{\mathcal{M}} : M^n \to M$ (obeys mod. of cont. $\delta(\varepsilon)$)
Rel. symbol	symbol R (arity n)	symbol R (arity n and mod. of cont. $\delta(\varepsilon)$)
Relations	$R^{\mathcal{M}} : M^n \to \{T,F\}$	$R^{\mathcal{M}} : M^n \to [0,1]$ (obeys mod. of cont. $\delta(\varepsilon)$)
Connectives	$\odot : \{T,F\}^n \to \{T,F\}$	$\odot : [0,1]^n \to [0,1]$ (continuous)
Sufficient con.	$\bot$, $\vee$, $\wedge$, $\to$	1, min, max, $\dot{-}$, $x \mapsto x/2$
Quantifiers	$\exists x\, \varphi(x)$, $\forall x\, \varphi(x)$	$\min_x \varphi(x)$, $\max_x \varphi(x)$
Formulas	$\varphi(\bar{x})$	$\varphi(\bar{x})$
Statements	Sentence: φ	Conditions: $[\varphi = 0]$, $[\varphi > 0]$
Evaluation	$\varphi^{\mathcal{M}} : M^n \to \{T,F\}$	$\varphi^{\mathcal{M}} : M^n \to [0,1]$
Satisfaction	$\mathcal{M} \models \varphi$ iff $\varphi^{\mathcal{M}} = T$	$\mathcal{M} \models [\varphi = 0]$ iff $\varphi^{\mathcal{M}} = 0$
Axioms/rules	complete, comp. list	complete, computable list
Provability	$\Sigma \vdash \varphi$	$\Sigma \vdash [\varphi > 0]$ (Σ set of conditions $[\psi = 0]$)

Main theorem

Theorem (R.)

Let $\mathcal{L}$ be a computable signature containing constants $\{c_n\}_{n\in\mathbb{N}}$.
Let Σ be a set of $\mathcal{L}$-conditions (using connectives 1, min, max, $\dot{-}$, $\cdot/2$).
Assume $(c_n^{\mathcal{M}})_{n\in\mathbb{N}}$ converges for all $\mathcal{M}\models\Sigma$.
Then there is a uniform rate of metastability for $(c_n^{\mathcal{M}})_{n\in\mathbb{N}}$ computable in Σ.

Proof.

- Fix $F\colon \mathbb{N}\to\mathbb{N}$ and $\varepsilon = 2^{-k}$, our goal is to find ℓ such that

$$\exists m < \ell\ \forall n,n' \in [m,F(m)]\ d^{\mathcal{M}}(c_n^{\mathcal{M}}, c_{n'}^{\mathcal{M}}) < \varepsilon.$$

- The quantifiers are bounded, so this can be written as the $\mathcal{L}$-condition

$$\underbrace{\max_{m<\ell}\ \min_{n,n'\in[m,F(m)]} \left(1/2^k \dot{-} d(c_n,c_{n'})\right)}_{\varphi_\ell} > 0.$$

Main theorem (cont.)

Proof.

- Fix $F\colon \mathbb{N} \to \mathbb{N}$ and $\varepsilon = 2^{-k}$, our goal is to find ℓ such that for all $\mathcal{M} \models \Sigma$,

$$\exists m < \ell \; \forall n, n' \in [m, F(m)] \; d^{\mathcal{M}}(c_n^{\mathcal{M}}, c_{n'}^{\mathcal{M}}) < \varepsilon.$$

- The quantifiers are bounded, so this can be written as the $\mathcal{L}$-condition

$$\underbrace{\max_{m<\ell} \min_{n,n' \in [m,F(m)]} \left(1/2^k \dot{-} d(c_n, c_{n'})\right)}_{\varphi_\ell} > 0.$$

- Notice φ_ℓ is computable in F, k, and ℓ.
- For all $\mathcal{M} \models \Sigma$, since $(c_n^{\mathcal{M}})_{n\in\mathbb{N}}$ converges, there exists ℓ s.t. $\mathcal{M} \models [\varphi_\ell > 0]$.
- By the compactness theorem, there is some ℓ such $\Sigma \models [\varphi_\ell > 0]$.
- By the completeness theorem, $\Sigma \vdash [\varphi_\ell > 0]$ for some ℓ.
- Compute ℓ by searching for a proof of $\Sigma \vdash [\varphi_\ell > 0]$ for some ℓ. □

Closing Thoughts

Summary

- There are many compatible logical tools for investigating theorems in analysis:
  - type theory and the dialectic interpretation,
  - ultraproducts, and
  - continuous logic.
- Usual (computability theoretic, proof theoretic, model theoretic) methods from first order logic extend nicely to continuous logic.
- There is a lot of potential to investigate computable continuous model theory; it is a nice merger of computable analysis and computable model theory.

- Can continuous logic be applied to proof mining in a useful way?
- Can logic methods be used to study other types of rates of convergence?

- **See my NERDS talk in two weeks for more details and examples.**

Thank You!

These slides will be available on my webpage:

http://www.personal.psu.edu/jmr71/

Or just Google™ me, "Jason Rute".

P.S. I am on the job market.