Randomness, Brownian Motion, Riesz Capacity, and Complexity

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Some of these results are joint work with Joseph Miller

Slides available at
www.personal.psu.edu/jmr71/

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This main idea of this talk

Randomness

True randomness

A point ω in a probability space $(\Omega, \mathbb{P})$ is **truly random** if ω satisfies every probability one property.

- Problem: True randomness does not exist!
- Solution: Restrict ourselves to computable probability one properties (algorithmic randomness).

Martin-Löf randomness

A point ω is **Martin-Löf random (MLR)** if there is no computable sequence of effectively open sets (U_n) such that $\omega \in \bigcap_{n \in \mathbb{N}} U_n$ and $\mathbf{P}(U_n) \leqslant 2^{-n}$ for all n.

- This definition holds on any computable probability space $(\Omega, \mathbf{P})$:
  - Ω is a computable Polish space (computable metric space).
  - $\mathbf{P}$ is a computable Borel probability measure on Ω.

Randomness for capacities

A new powerful new tool

- All measures are additive:

$$\mu(A \cup B) = \mu(A) + \mu(B) \quad (A \text{ and } B \text{ disjoint})$$

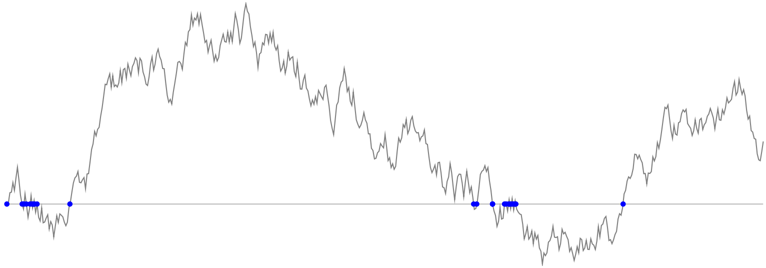
- **Capacities** are like measures except subadditive

$$\mu(A \cup B) \leqslant \mu(A) + \mu(B) \quad (A \text{ and } B \text{ disjoint})$$

- Martin-Löf randomness can be extended to "computable capacities."
  - Using the exact same definition!
- This provides a **unified framework** for many ideas in randomness.

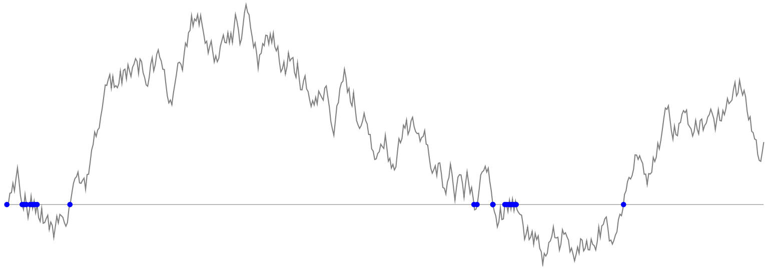
What are the zeros of MLR Brownian motion?

Brownian motion



- **Brownian motion** B is a "continuous time random walk."
- Brownian motion paths are continuous, start at zero, and wiggle a lot.
- Brownian motion is given by the computable probability space $(C([0,1]), \mathbf{P}_{\mathsf{B}})$ where $\mathbf{P}_{\mathsf{B}}$ is the Wiener probability measure.
- A **Martin-Löf random Brownian motion path** is a MLR for $(C([0,1]), \mathbf{P}_{\mathsf{B}})$.

Zeros of Brownian motion



- The **zero set** of a Brownian motion path B is $Z_B = \{t : B(t) = 0\}$.
- Almost surely the zero sets are
  - Closed. No isolated points. Hausdorff dimension 1/2.
- (A/B/S) The zero sets of MLR Brownian motion paths are
  - Closed. No isolated points. Hausdorff dimension 1/2.

Zeros of MLR Brownian motion

Question (Allen/Bienvenu/Slaman)

Which times $t \in [0,1]$ are the zero times of a MLR Brownian motion?

For a real $t \in [0,1]$, with binary representation $T \in 2^\omega$, the **constructive dimension** of t is

$$\mathrm{cdim}(t) = \liminf_n \frac{K(T \upharpoonright n)}{n} = \liminf_n \frac{KM(T \upharpoonright n)}{n}.$$

Theorem (Allen/Bienvenu/Slaman)

For $t > 0$:

- *If cdim$(t) > 1/2$, then t is a zero of some MLR Brownian motion path.*
- *If cdim$(t) < 1/2$, then t is not a zero of any MLR Brownian motion path.*
- ***If cdim$(t) = 1/2$, then it is inconclusive.***

Characterization of the zeros of MLR Brownian motion

- Trivially $t = 0$ is a zero time.
- So from now on work in $[a, b]$ for fixed rationals $0 < a < b \leqslant 1$.

Theorem (R.; Miller/R.)

For $t \in [a, b]$ (with binary representation $T \in 2^\omega$), the following are equivalent.

1. *t is a zero of a MLR Brownian Motion path.*
2. *t is a member of a MLR compact set for the probability distribution $\mathbf{P}_{Z_B}$.*
3. *t is MLR for the* ***intersection capacity*** $\mathbf{T}_{Z_B}$.
4. *t is MLR for the $1/2$-**Riesz capacity*** $\mathsf{Cap}_{1/2}$.
5. *t is MLR for the $1/2$-**Riesz capacity*** $\mathsf{C}_{1/2}$.
6. *t is $1/2$-energy random (i.e. t is MLR for a measure with finite $1/2$-energy).*
7. *T is $1/2$-energy random (i.e. T is MLR for a measure with finite $1/2$-energy).*
8. *T satisfies $\sum_n 2^{n/2 - KM(T \upharpoonright n)} < \infty$.*

Random compact sets

Zeros as random compact sets

- Brownian motion B is a **continuous-function-valued random variable**.
- $\mathsf{Z}_\mathsf{B} = \{t \in [a,b] : \mathsf{B}(t) = 0\}$ is a **compact-set-valued random variable**,
  - a.k.a. a **random compact set** or **random closed set**.
- $\mathcal{K}([a,b])$, the space of compact sets on $[a,b]$ is a computable Polish space (under the Hausdorff metric).
- $(\mathcal{K}([a,b]), \mathbf{P}_{\mathsf{Z}_\mathsf{B}})$ is a computable probability space.
- Hence, one can refer to **$\mathbf{P}_{\mathsf{Z}_\mathsf{B}}$-Martin-Löf random compact sets**.

From MLR Brownian motion paths to MLR compact sets

Theorem

Z is the zero set of a MLR Brownian motion iff Z is a $\mathbf{P}_{\mathsf{Z_B}}$-MLR compact set.

Proof.

The map

$$B \quad \mapsto \quad Z_B := \{t \in [a,b] : B(t) = 0\}$$

is $\mathbf{P}_{\mathsf{B}}$-almost-surely computable and measure-preserving as a map of type

$$(C([0,1]), \mathbf{P}_{\mathsf{B}}) \quad \to \quad (\mathcal{K}([a,b]), \mathbf{P}_{\mathsf{Z_B}}).$$

- By randomness preservation: If B is $\mathbf{P}_{\mathsf{B}}$-MLR, then Z_B is $\mathbf{P}_{\mathsf{Z_B}}$-MLR.
- By no-randomness-from-nothing: If Z is MLR, then $\exists$ MLR B s.t. $Z = Z_B$.

□

Characterization of the zeros of MLR Brownian motion

...as members of MLR closed sets

Theorem (R.)

For $t \in [a,b]$, the following are equivalent.

1. *t is a zero of a MLR Brownian Motion path.*
2. ***t is a member of a MLR compact set for the probability distribution $\mathbf{P}_{Z_B}$.***

Intersection Capacities

Intersection capacities

- Let X be a random compact set (such as Z_B).
- The probability that a (deterministic) set A intersects X is known as the **intersection capacity** of A

$$\mathbf{T}_\mathsf{X}(A) := \mathbf{P}\{A \cap \mathsf{X} \neq \varnothing\}.$$

- If the probability distribution $\mathbf{P}_\mathsf{X}$ is computable, then $\mathbf{T}_\mathsf{X}$ is a computable capacity.

Theorem (R.)

The following are equivalent.

1. *x is a member of some ML random compact set for the prob. distribution $\mathbf{P}_\mathsf{X}$.*
2. *x is ML random for the intersection capacity $\mathbf{T}_\mathsf{X}$.*

Characterization of the zeros of MLR Brownian motion

...as MLRs for intersection capacity

Theorem (R.)

For $t \in [a,b]$, the following are equivalent.

1. *t is a zero of a MLR Brownian Motion path.*
2. *t is a member of a MLR compact set for the probability distribution $\mathbf{P}_{\mathsf{Z}_\mathsf{B}}$.*
3. ***t is MLR for the intersection capacity $\mathbf{T}_{\mathsf{Z}_\mathsf{B}}$.***

Potential theory and Riesz capacity

Potential theory

For $s > 0$,

- The ***s*-dimensional energy** of a measure $\mu \in \mathcal{M}(\mathbb{R})$ is

$$\mathsf{Energy}_s(\mu) = \int_{\mathbb{R}} \int_{\mathbb{R}} |x-y|^{-s} \, d\mu(x) d\mu(y).$$

- ***s*-Riesz capacity** (version 1)

$$\mathsf{Cap}_s(A) := \sup \left\{ \frac{1}{\mathsf{Energy}_s(\mu)} \,\middle|\, \mu \in \mathcal{M}_1(A) \right\}.$$

Theorem (Kakutani)

$\mathbf{T}_{\mathsf{Z_B}}(A) =^{\times} \mathsf{Cap}_{1/2}(A) \quad$ for measurable $A \subseteq [a,b]$ $(0 < a < b)$.

Corollary (R.)

The following are equivalent for computable $s > 0$.

- *$t \in [a,b]$ is MLR for the intersection capacity $\mathbf{T}_{\mathsf{Z_B}}$.*
- *$t \in [a,b]$ is MLR for the $1/2$-Riesz capacity $\mathsf{Cap}_{1/2}$.*

Characterization of the zeros of MLR Brownian motion

...as MLRs for 1/2-Riesz capacity

Theorem (R.)

For $t \in [a,b]$, the following are equivalent.

1. *t is a zero of a MLR Brownian Motion path.*
2. *t is a member of a MLR compact set for the probability distribution $\mathbf{P}_{Z_B}$.*
3. *t is MLR for the intersection capacity $\mathbf{T}_{Z_B}$.*
4. ***t is MLR for the $1/2$-Riesz capacity $\mathsf{Cap}_{1/2}$.***

Other Riesz capacity

- **s-Riesz capacity** (version 2)

$$\mathsf{C}_s(A) := \sup\{\mu(A) \mid \mu \in \mathcal{M}(\mathbb{R}),\ \mathsf{Energy}_s(\mu) \leqslant 1\}.$$

Theorem

$\mathsf{Cap}_s = (\mathsf{C}_s)^2$.

Corollary

The following are equivalent for computable $s > 0$.

- *$x \in \mathbb{R}$ is MLR for the s-Riesz capacity Cap_s.*
- *$x \in \mathbb{R}$ is MLR for the s-Riesz capacity C_s.*

Characterization of the zeros of MLR Brownian motion

...as MLRs for 1/2-Riesz capacity

Theorem (R.)

For $t \in [a,b]$, the following are equivalent.

1. *t is a zero of a MLR Brownian Motion path.*
2. *t is a member of a MLR compact set for the probability distribution $\mathbf{P}_{\mathsf{Z_B}}$.*
3. *t is MLR for the intersection capacity $\mathbf{T}_{\mathsf{Z_B}}$.*
4. *t is MLR for the 1/2-Riesz capacity $\mathsf{Cap}_{1/2}$.*
5. ***t is MLR for the 1/2-Riesz capacity $\mathsf{C}_{1/2}$.***

Energy randomness and upper-envelopes

Energy randomness

- Recall,

$$\mathsf{Energy}_s(\mu) = \int_{\mathbb{R}} \int_{\mathbb{R}} |x-y|^{-s} \, d\mu(x) d\mu(y).$$

- MLR also extends to noncomputable measures (basically, use the measure as an oracle...with some subtleties)

Definition (Diamondstone/Kjos-Hanssen)

$x \in \mathbb{R}$ is ***s*-energy random** if

- x is MLR for some (not necessarily computable) probability measure $\mu \in \mathcal{M}_1(\mathbb{R})$ with $\mathsf{Energy}_s(\mu) < \infty$.
- (or equivalently) x is MLR for some (not necessarily computable) measure $\mu \in \mathcal{M}(\mathbb{R})$ with $\mathsf{Energy}_s(\mu) \leqslant 1$.

Upper envelope capacities

- $x \in \mathbb{R}$ is s-energy random iff x is MLR for some measure μ in the set

$$K_s := \{\mu \mid \mu \in \mathcal{M}(\mathbb{R}),\ \mathsf{Energy}_s(\mu) \leqslant 1\}.$$

- This compact set K_s of measures is computable in $\mathcal{K}(\mathcal{M}(\mathbb{R}))$.

- Recall

$$\mathsf{C}_s(A) := \sup\left\{\mu(A) \,\middle|\, \underbrace{\mu \in \mathcal{M}(\mathbb{R}),\ \mathsf{Energy}_s(\mu) \leqslant 1}_{\mu \in K_s}\right\}.$$

- This is an **upper envelope capacity**. (C_s is the upper envelope of K_s).

Randomness for a compact set of measures

Theorem (Basically Bienvenu/Hoyrup/Gács/Rojas/Shen)

Assume $\mathsf{C}_K(A) = \sup_{\mu \in K} \mu(A)$ *for some compact set* K *of measures computable in* $\mathcal{K}(\mathcal{M}(\mathbb{R}))$*, then the following are equivalent.*

1. x *is MLR for the upper envelope capacity* C_K.
2. x *is MLR for the some measure* $\mu \in K$.

Corollary

The following are equivalent for computable $s > 0$.

1. t *is MLR for the s-Riesz capacity* C_s.
2. t *is MLR for some measure* $\mu \in \mathcal{M}(\mathbb{R})$ *such that* $\mathsf{Energy}_s(\mu) \leqslant 1$.
3. t *is MLR for some prob. measure* $\mu \in \mathcal{M}_1(\mathbb{R})$ *such that* $\mathsf{Energy}_s(\mu) < \infty$.
4. t *is s-energy random.*

Characterization of the zeros of MLR Brownian motion

...as 1/2-energy randoms

Theorem (R.)

For $t \in [a,b]$, the following are equivalent.

1. *t is a zero of a MLR Brownian Motion path.*
2. *t is a member of a MLR compact set for the probability distribution $\mathbf{P}_{\mathsf{Z_B}}$.*
3. *t is MLR for the intersection capacity $\mathbf{T}_{\mathsf{Z_B}}$.*
4. *t is MLR for the $1/2$-Riesz capacity $\mathsf{Cap}_{1/2}$.*
5. *t is MLR for the $1/2$-Riesz capacity $\mathsf{C}_{1/2}$.*
6. ***t is $1/2$-energy random (i.e. t is MLR for a measure with finite $1/2$-energy).***

Potential theory on Cantor space

Potential theory on 2^ω

For $s > 0$,

- The ***s*-dimensional potential** of a measure $\mu \in \mathcal{M}(2^\omega)$ is the function

$$\mathsf{Potential}_s(\mu)(X) = \sum_n 2^{sn}\mu[X \upharpoonright n] \qquad X \in 2^\omega.$$

- The ***s*-dimensional energy** of a measure $\mu \in \mathcal{M}(2^\omega)$ is

$$\mathsf{Energy}_s(\mu) = \int_{2^\omega} \mathsf{Potential}_s(\mu)(X)\, d\mu(X) = \sum_{\sigma \in 2^{<\omega}} 2^{s|\sigma|}(\mu[\sigma])^2.$$

- $X \in 2^\omega$ is ***s*-energy random** if X is MLR for some probability measure μ with finite s-energy.

Theorem [R., Miller/R.]

For $x \in [0,1]$ with binary representation $X \in 2^\omega$, the following are equivalent.

1. x is s-energy random.
2. X is s-energy random.

Characterization of the zeros of MLR Brownian motion

...as 1/2-energy randoms in 2^ω

Theorem (R.; Miller/R.)

For $t \in [a,b]$ (with binary representation $T \in 2^\omega$), the following are equivalent.

1. *t is a zero of a MLR Brownian Motion path.*
2. *t is a member of a MLR compact set for the probability distribution $\mathbf{P}_{Z_B}$.*
3. *t is MLR for the intersection capacity $\mathbf{T}_{Z_B}$.*
4. *t is MLR for the $1/2$-Riesz capacity $\mathsf{Cap}_{1/2}$.*
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6. *t is $1/2$-energy random (i.e. t is MLR for a measure with finite $1/2$-energy).*
7. ***T is $1/2$-energy random (i.e. T is MLR for a measure with finite $1/2$-energy).***

Energy randomness via a priori complexity

Joint work with Joseph Miller

Semimeasures and a priori complexity

- A (continuous) semimeasure is a function $\rho\colon 2^{<\omega} \to [0,1]$ such that

$$\rho(\sigma 0) + \rho(\sigma 1) \leqslant \rho(\sigma) \qquad (\sigma \in 2^{<\omega}).$$

- There is a **universal lower-semicomputable semimeasure** $\mathbf{M}$ such that $\rho \leqslant^{\times} \mathbf{M}$ for all lower-semicomputable semimeasures ρ.

- The **a priori complexity** of a string σ is given by

$$KM(\sigma) := -\log_2 \mathbf{M}(\sigma).$$

- $KM(\sigma)$ is a measure of the computational complexity of σ,
  - similar to (but not the same as) prefix-free Kolmogorov complexity K.

- $X \in 2^{\omega}$ is MLR for a computable measure μ iff

$$KM(X \upharpoonright n) \geqslant^{+} -\log_2 \mu[X \upharpoonright n].$$

Complexity characterization of s-energy randomness

Theorem (Miller/R.)

The following are equivalent for $X \in 2^\omega$ and computable $s > 0$.

- *X is s-energy random.*
- $\sum_n 2^{sn - KM(X \upharpoonright n)} < \infty$.

- Some people like this result because:
  - It puts energy randomness in terms they understand.
  - (Read: It gets rid of all the analysis!)
  - It looks a lot like the ample excess lemma of Gács, Miller/Yu

$$X \text{ is MLR on Lebesgue measure} \quad \text{iff} \quad \sum_n 2^{n - K(X \upharpoonright n)} < \infty.$$

- I like this result because:
  - It is related to strong s-randomness, $KM(X \upharpoonright n) \geqslant^+ sn$.
  - It is equivalent to saying

$$\mathsf{Potential}_s(\mathbf{M})(X) = \sum_n 2^{sn} \mathbf{M}(X \upharpoonright n) < \infty.$$

Characterization of the zeros of MLR Brownian motion

...via a priori complexity

Theorem (R.; Miller/R.)

For $t \in [a,b]$ (with binary representation $T \in 2^\omega$), the following are equivalent.

1. *t is a zero of a MLR Brownian Motion path.*
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3. *t is MLR for the intersection capacity $\mathbf{T}_{\mathsf{Z_B}}$.*
4. *t is MLR for the $1/2$-Riesz capacity $\mathsf{Cap}_{1/2}$.*
5. *t is MLR for the $1/2$-Riesz capacity $\mathsf{C}_{1/2}$.*
6. *t is $1/2$-energy random (i.e. t is MLR for a measure with finite $1/2$-energy).*
7. *T is $1/2$-energy random (i.e. T is MLR for a measure with finite $1/2$-energy).*
8. ***T satisfies $\sum_n 2^{n/2 - KM(T\upharpoonright n)} < \infty$.***

This implies Allen/Bienvenu/Slaman's result about constructive dimension

$$\mathrm{cdim}(T) = \liminf_n \frac{KM(T \upharpoonright n)}{n}.$$

Other applications

Other random compact sets

The same ideas can be used to give characterizations of

- Points visited by a MLR d-dimensional Brownian motion.
- Points in the intersection of n independent MLR 2-dimensional BM.
- Points visited n-times by the same MLR 2-dimensional Brownian motion.
- Members of the random closed sets of Barmpalias/Brodhead/Cenzer/Dashi/Weber.
- And more...

In each case, the key is to find an estimate of the intersection capacity.

Examples of randomness for capacities

- strong s-randomness (resp. strong f-randomness) .... (Reimann and many others)
  - $=$ randomness on s-dimensional (resp. f-weighted) Hausdorff capacity
- s-energy randomness .... (Diamondstone/Kjos-Hanssen)
  - $=$ randomness on s-dimensional Riesz capacity
- MLR for a class of measures .... (Bienvenu/Gács/Hoyrup/Rojas/Shen)
  - $=$ randomness on the corresponding upper envelope capacity
- members of a MLR closed set
  - MLR closed sets............... (Barmpalias/Brodhead/Cenzer/Dashti/Weber)
  - zeros of MLR Brownian motion .... (Kjos-Hanssen/Nerode and A/B/S)
  - image of MLR n-dim. Brownian motion .... (Allen/Bienvenu/Slaman)
  - double points of MLR planar BM.................. (Allen/Bienvenu/Slaman)
  - $=$ randomness on the corresponding intersection capacity
- (Unfinished work) Lebesgue points of all computable Sobolev $W^{n,p}$ functions
  - $=$ (some sort of) Schnorr randomness on n,p-Bessel capacity

Summary

Randomness for capacities...

- provides a unified framework for concepts in randomness
- helps us to prove new results
- simplifies the proofs of old results
- gives us 60-years-worth of theorems in capacity theory to draw from!

Closing Thoughts

Thank You!

These slides will be available on my webpage:

http://www.personal.psu.edu/jmr71/

Or just Google™ me, "Jason Rute".