THE POWER OF UNIFORMITY IN ALGORITHMIC RANDOMNESS

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1. Introduction to algorithmic randomness

Consider an infinite binary string $x \in \{0,1\}^{\mathbb{N}}$. For example,

$$x = 0101010101010101010 \ldots \quad \text{or}$$
$$y = 01011110001111001001 \ldots$$

Algorithmic randomness gives us a robust way to say which strings are *random* and which are not. The idea is that the random x are those which pass all computable statistical tests (computable $\Rightarrow$ finitely many tests).

Functions.

(1) A (TOTAL) COMPUTABLE FUNCTION $f\colon \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ is a function such that $f(x)$ is computable uniformly from x.

(2) A LOWER SEMICOMPUTABLE FUNCTION $T\colon \{0,1\}^{\mathbb{N}} \to [0,\infty]$ is a function of the form

$$T = \sup_n f_n$$

for a uniformly computable sequence of total computable functions $f_n\colon \{0,1\}^{\mathbb{N}} \to [0,\infty)$.

Measures.

(1) Let $\mathcal{M}_1 = \mathcal{M}_1(\{0,1\}^{\mathbb{N}})$ be the space of (Borel) probability measures on $\{0,1\}^{\mathbb{N}}$.
(2) Each measure $\mu \in \mathcal{M}_1$ is determined by $\mu([\sigma])$ for all $\sigma \in \{0,1\}^*$.
(3) A COMPUTABLE PROBABILITY MEASURE μ is one where $\sigma \mapsto \mu([\sigma])$ is computable.
(4) Let λ denote the FAIR COIN MEASURE, $\lambda([\sigma]) = 2^{-|\sigma|}$.

Randomness. Let μ be a *computable measure.*

(1) Tests
  (a) A MARTIN-LÖF (INTEGRAL) TEST w.r.t. μ is a lower semicomputable function $T\colon \{0,1\}^{\mathbb{N}} \to [0,\infty]$ such that $\int T(x)\, d\mu(x) \le 1$ (or $\le \infty$).
  (b) A SCHNORR (INTEGRAL) TEST is a lower semicomputable function $T\colon \{0,1\}^{\mathbb{N}} \to [0,\infty]$ such that $\int T(x)\, d\mu(x) = 1$ (or computable).
(2) Randoms
  (a) $x \in \{0,1\}^{\mathbb{N}}$ is MARTIN-LÖF RANDOM w.r.t. μ ($x \in MLR_\mu$) if $T(x) < \infty$ for all Martin-Löf integral tests T.
  (b) $x \in \{0,1\}^{\mathbb{N}}$ is SCHNORR RANDOM w.r.t. μ ($x \in SR_\mu$) if $T(x) < \infty$ for all Schnorr integral tests T.
(3) Relative randomness (classical approach)
  (a) $x \in \{0,1\}^{\mathbb{N}}$ is MARTIN-LÖF RANDOM (W.R.T. μ) RELATIVE TO x ($x \in MLR^y_\mu$) if $T(x) < \infty$ for all Martin-Löf integral tests T *computable from* y.
  (b) $x \in \{0,1\}^{\mathbb{N}}$ is SCHNORR RANDOM (W.R.T. μ) RELATIVE TO y ($X \in SR^y_\mu$) if $T(x) < \infty$ for all Schnorr integral tests T *computable from* y.

Observations.

- $SR \supset MLR$.
- Schnorr randomness is a *more effective* version of Martin-Löf randomness.

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2. Randomness v.s. computable analysis

Characterizations of Martin-Löf randomness. *The Martin-Löf randoms (w.r.t. λ) are exactly those x which satisfy one/all of the following.*

(1) ***Ergodic theorem (V'yugen; Franklin/Towsner).*** *For a.e. computable, measure preserving, maps $\varphi\colon \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$, and for all continuous functions f,*

$$\frac{1}{n}\sum_{k<n} f(\varphi^k(x)) \quad \text{converges.}$$

(2) ***Differentiation theorem on measures (Rute).*** *For all computable measures μ,*

$$\frac{\mu(B_r(x))}{\lambda(B_r(x))} \quad \text{converges as } r \to 0.$$

(3) ***Lebesgue's theorem (Brattka/Miller/Nies).*** *For all computable and absolutely continuous functions $f\colon [0,1] \to \mathbb{R}$, then f is differentiable at x.*

(4) ***Martingale convergence (Rute).*** *For all $L^1(\lambda)$-computable martingales M_n,*

$$M_n(x) \quad \text{converges as } n \to \infty.$$

(5) *...*

Characterizations of Schnorr randomness. *The Schnorr randoms (w.r.t. λ) are exactly those x which satisfy one/all of the following.*

(1) ***Normality (Schnorr).*** *For every map $\varphi\colon \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ which is*
  (a) *a.e. computable, i.e. $\lambda(dom(\varphi)) = 1$, and*
  (b) *measure preserving, i.e. $\lambda(\varphi^{-1}([\sigma])) = \lambda(\sigma)$,*
  then $\varphi(x)$ is normal, i.e. $\frac{1}{n}\sum_k x_n = 1/2$.

(2) ***Ergodic theorem (Gács/Hoyrup/Rojas).*** *For a.e. computable, measure preserving,* ergodic *maps $\varphi\colon \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$, and for all continuous functions f,*

$$\frac{1}{n}\sum_{k<n} f(\varphi^k(x)) \to \int f\, d\lambda.$$

(3) ***Lebesgue differentiation theorem (Rute; Pathak/Rojas/Simpson).*** *For all $L^1(\lambda)$-computable functions f,*

$$\frac{1}{\lambda(B_r(x))}\int_{B_r(x)} f\, d\lambda \quad \text{converges as } r \to 0.$$

(4) ***Differentiation theorem on measures (Rute).*** *For all μ effectively absolutely continuous to λ,*

$$\frac{\mu(B_r(x))}{\lambda(B_r(x))} \quad \text{converges as } r \to 0.$$

(5) ***Lebesgue's theorem (Rute; Freer/Kjos-Hanssen/Nies/Stephan).*** *For all effectively absolutely continuous functions $f\colon [0,1] \to \mathbb{R}$, then f is differentiable at x.*

(6) ***Martingale convergence (Rute).*** *For all nonnegative $L^1(\lambda)$-computable martingales M_n with L^1-computable limit,*

$$M_n(x) \quad \text{converges as } n \to \infty.$$

(7) *...*

Computable analysis can benefit randomness by ....

(1) characterizing randomness notions (as above),
(2) providing justification that these randomness notions are the correct ones,
(3) providing applications of randomness,
(4) **providing insight and new motivations in randomness**
(analysis gives the words and definitions to explain what is really going on),
(5) **providing tools to prove theorems in randomness**
(this is similar to the powerful applications of analysis to discrete math and number theory).

3. Van Lambalgen's theorem

Write (x, y) to mean $x \oplus y$. (This is justified since $(x, y) \mapsto x \oplus y$ is a computable isomorphism.) Write SR and MLR for SR_λ and MLR_λ.

Theorem (van Lambalgen)**.** *$(x, y) \in MLR$ if and only if $x \in MLR$ and $y \in MLR^x$.*

Theorem (Merkle, Miller, Nies, Reimann, Stephan 2006; Yu 2007)**.** *Exists $(a, b) \in SR$ such that $a \notin SR^b$ and $b \notin SR^a$.*

Proof (one-liner due to Kjos-Hanssen). There is a Schnorr random of minimal Turning degree. Hence each half computes the other! □

Theorem (Franklin/Stephan 2011; Rute/Miyabe 2013; following incorrect folklore)**.** *If $a \in SR$ and $b \in SR^a$, then $(a, b) \in SR$.*

4. Uniformly relativized Schnorr randomness

Definition. A uniform Schnorr integral test is a family $\{T^x\}_{x\in\{0,1\}^{\mathbb{N}}}$ such that

- $T^x : \{0, 1\}^{\mathbb{N}} \to [0, \infty]$ is lower semicomputable uniformly in x. (Equivalently, $T^x(y) = T'(x, y)$ for a some lower semicomputable $T' : \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}} \to [0, \infty]$.)
- $\int T^x(y)\, d\lambda(y) = 1$ for all x.

Definition (Franklin/Stephan 2010 "truth-table Schnorr randomness")**.** $a \in \{0, 1\}^{\mathbb{N}}$ is Schnorr random uniformly relative to b ($a \in uSR^b$) if $T^b(a) < \infty$ for all uniform Schnorr tests $\{T^x\}$.

Remark. This is not a new randomness notion! Its a new (old!) notion of relativization. $SR = SR^\emptyset = uSR^\emptyset$. Also, for Martin-Löf randomness, $uMLR^b = MLR^b$.

Theorem (Miyabe 2011 (⇒); Miyabe/Rute 2013 (⇐))**.** *$(a, b) \in SR$ if and only if $a \in SR$ and $b \in uSR^a$.*

5. Schnorr randomness for non-computable measures

Definition (Levin)**.** A uniform (Martin-Löf integral) test is a family $\{T_\mu\}_{\mu\in\mathcal{M}_1}$ such that

- $T_\mu : \{0, 1\}^{\mathbb{N}} \to [0, \infty]$ is lower semicomputable uniformly in μ.
- $\int T_\mu(y)\, d\mu(y) < 1$ for all x.

Definition (Levin)**.** $a \in \{0, 1\}^{\mathbb{N}}$ is Martin-Löf random w.r.t. μ_0 ($a \in ML_{\mu_0}$) if $T_{\mu_0}(a) < \infty$ for all uniform tests $\{T_\mu\}_{\mu\in\mathcal{M}_1}$.

Definition (Rute)**.** A uniform Schnorr integral test is a family $\{T_\mu\}_{\mu\in\mathcal{M}_1}$ such that

- $T_\mu : \{0, 1\}^{\mathbb{N}} \to [0, \infty]$ is lower semicomputable uniformly in μ.
- $\int T_\mu(y)\, d\mu(y) = 1$ for all x.

Definition (Rute)**.** $a \in \{0, 1\}^{\mathbb{N}}$ is Schnorr random w.r.t. μ_0 ($a \in uSR_{\mu_0}$) if $T_{\mu_0}(a) < \infty$ for all uniform tests $\{T_\mu\}_{\mu\in\mathcal{M}_1}$.

Definition (Rute)**.** $a \in \{0, 1\}^{\mathbb{N}}$ is Schnorr random w.r.t. μ_0 relative to b ($a \in uSR^b_{\mu_0}$) if $T^b_{\mu_0}(a) < \infty$ for all uniform tests $\{T^y_\mu\}_{\mu\in\mathcal{M}_1, y\in Y}$ (where Y is some computable metric space).

6. Main Result

Theorem (Bienvenu et al.; c.f. Day/Reimann)**.** *For (noncomputable) probability measures μ and ν,*

$$(a, b) \in MLR_{\mu\otimes\nu} \quad \textit{iff} \quad a \in MLR^\nu_\mu \text{ and } b \in MLR^{\mu,a}_\nu.$$

Main Theorem (Rute, In preparation)**.** *For (noncomputable) probability measures μ and ν,*

$$(a, b) \in uSR_{\mu\otimes\nu} \quad \textit{iff} \quad a \in uSR^\nu_\mu \text{ and } b \in uSR^{\mu,a}_\nu.$$

7. Our analytic toolkit

Approximating Lower Semicomputable Functions Lemma.

(1) *A function T is lower semicomputable iff $T = \sup_{t>1} f_t$ where $f_t(x)$ is computable from t and x.*
(2) *If $\int T\, d\mu = 1$ (and $T(x) > 0$ for all x), then can make $\int f_t\, d\mu = t$.*

Proof. There is a increasing sequence of computable functions g_n such that $T = \sup_n g_n$. Let

$$f_t(x) = g_{\lfloor t \rfloor}(x) + (t - \lfloor t \rfloor)\left(g_{\lfloor t \rfloor + 1}(x) - g_{\lfloor t \rfloor}(x)\right).$$

For the second part, just reparameterize f_t to match the integral (modulo some technical details.) □

Effective Tiestz Extension Theorem. *If X is a computable metric space, $K \subseteq X$ is effectively closed, and $f\colon K \to \mathbb{R}$ is computable, then we can effectively extended f to $g\colon X \to \mathbb{R}$.*

Test Extension Lemma. *A uniform Schnorr test $\{S_\mu\}_{\mu \in K}$ restricted to an effectively closed set K of measures, can be extended to a uniform Schnorr test $\{S_\mu\}_{\mu \in \mathcal{M}_1}$.*

Proof. By the Approximating Lower Semicomputable Functions Lemma, we may assume there is a computable family $\{f_{\mu;t}\}_{\mu \in K, t \in [0,1)}$ such that for $\mu \in K$ we have $S_\mu = \sup f_{\mu;t}$ and $\int f_{\mu;t}\, d\mu = t$. Then use the Effective Tiestz Extension Theorem to extend $\{f_{\mu;t}\}_{\mu \in K}$ to $\{g_{\mu;t}\}_{\mu \in \mathcal{M}_1}$ outside of K. Last, normalize $g_{\mu;t}$ by replacing it with $g_{\mu;t} / \int g_{\mu;t}$ (modulo some precautions so as to not divide by zero). □

Corollary. *The definition of Schnorr randomness w.r.t. arbitrary (not-necessarily-computable) μ extends the usual definition w.r.t. computable μ.*

Proof. By the Test Extension Lemma, any Schnorr integral test on a computable measure μ can be extended to all measures. □

Effective Luzin's Theorem (Miyabe/Rute; also c.f. layerwise computable functions)**.** *If T is lower semicomputable, $\int T\, d\mu = 1$, and x is Schnorr random, then there is some computable $h\colon \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ such that $h \le T$ and $h(x) = T(x)$.*

8. Proof of the "easy" direction

"Easy" direction. *If $(a, b) \in uSR_{\nu_0 \otimes \mu_0}$*, then $b \in uSR_{\nu_0}^{\mu_0, a}$.

Proof. Prove the contrapositive. Assume, $b \notin uSR_{\nu_0}^{\mu_0, a}$. Then, there is a uniform Schnorr test $\{T_\mu^{x,\nu}\}_{x \in \{0,1\}^{\mathbb{N}}, \nu \in \mathcal{M}_1, \mu \in \mathcal{M}_1}$ such that $T_{\mu_0}^{a,\nu_0}(b) = \infty$.

Consider the family of functions $\{S_\rho\}_{\rho \in P}$ indexed by the product measures

$$P = \{\mu \otimes \nu \mid \mu, \nu \in \mathcal{M}_1\},$$

and where $S_{\mu \otimes \nu}$ is given by

$$S_{\mu \otimes \nu}(x, y) := T_\mu^{x,\nu}(y).$$

We have that $\{S_\rho\}_{\rho \in P}$ is a uniform Schnorr test, since $S_{\mu \otimes \nu}(x, y)$ is lower semicomputable in $\mu \otimes \nu$, x, y; and since

$$\iint S(x, y)\, d(\mu \otimes \nu)(x, y) = \int \left(\int T_\mu^{x,\nu}(y)\, d\nu(y) \right) d\mu(x) = \int 1\, d\mu = 1.$$

Since P is effectively closed, by our Test Extension Lemma, we can extend $\{S_\rho\}_{\rho \in P}$ to a uniform Schnorr test.

The rest follows since $S_{\mu_0 \otimes \nu_0}(a, b) = \infty$, and therefore $(a, b) \notin uSR_{\nu_0 \otimes \mu_0}$. □

9. Proof of the hard direction.

"Hard" direction. *If* $a \in uSR^{\nu_0}_{\mu_0}$ and $b \in uSR^{\mu_0,a}_{\nu_0}$, then $(a,b) \in uSR_{\nu_0 \otimes \mu_0}$.

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Proof. Assume $(a,b) \notin uSR_{\mu_0 \otimes \nu_0}$ and $a \in uSR^{\nu_0}_{\mu_0}$. Then there is a uniform Schnorr integral test $\{T_\rho\}_{\rho \in \mathcal{M}_1}$ such that $T_{\mu_0 \otimes \nu_0}(a,b) = \infty$.

Consider the family $\{S^{x,\nu}_\mu\}_{x \in \{0,1\}^\mathbb{N}, \nu \in \mathcal{M}_1, \mu \in \mathcal{M}_1}$ of lower semicomputable functions given by

$$S^{x,\nu}_\mu(y) = T_{\mu \otimes \nu}(x,y).$$

However, $\{S^{x,\nu}_\mu\}_{x \in \{0,1\}^\mathbb{N}, \nu \in \mathcal{M}_1, \mu \in \mathcal{M}_1}$ may not be a uniform test since, it is possible that $\int S^{x,\nu}_\mu(y)\, dy = \infty$ some x (albeit a μ-measure zero set for each μ). To fix this, consider the this family of functions

$$I^\nu_\mu(x) = \int S^{x,\nu}_\mu(y)\, d\nu(y) = \int T_{\mu \otimes \nu}(x,y)\, d\nu(y).$$

Then $I^\nu_\mu(x)$ is lower semicomputable in ν, μ, x, and

$$\int I^\nu_\mu(x)\, d\mu(x) = \iint T_{\mu \otimes \nu}(x,y)\, d(\mu \otimes \nu)(x,y) = 1.$$

Since $a \in uSR^{\nu_0}_{\mu_0}$, we have that $\int S^{x,\nu}_\mu(y)\, d\nu(y) = I^{\nu_0}_{\mu_0}(a) < \infty$. But we need more.

By our Effective Luzin's Theorem, there is some computable $h^\nu_\mu \colon \{0,1\}^\mathbb{N} \to \mathbb{R}$ such that $h \le I^\nu_\mu$ and $h^{\nu_0}_{\mu_0}(a) = I^{\nu_0}_{\mu_0}(a)$. Let $\{\hat{S}^{x,\nu}_\mu\}$ be the family of lower semicomputable functions which equals $S^{x,\nu}_\mu(y)$ except that we "cut off" the enumeration so that $\int \hat{S}^{x,\nu}_\mu(y)\, d\nu(y) = h^\nu_\mu(x)$.

Formally, let $\{f^{x,\nu}_{\mu;(t)}\}$ be the family of continuous functions computable in x, ν, μ, t such that $S^{x,\nu}_\mu = \sup_{t<1} f^{x,\nu}_{\mu;(t)}$. Then for x, μ, ν, let

$$\hat{S}^{x,\nu}_\mu := \sup \left\{ f^{x,\nu}_{\mu;(t)} \,\middle|\, t \le 1 \text{ and } \int f^{x,\nu}_{\mu;(t)}\, d\nu \le h^\nu_\mu(x) \right\}.$$

Then $\int \hat{S}^{x,\nu}_\mu d\nu = h^\nu_\mu(x)$.

Then $\{\hat{S}^{x,\nu}_\mu\}$ is a uniform Schnorr test and $\hat{S}^{a,\nu_0}_{\mu_0} = S^a$. Therefore $\hat{S}^{a,\nu_0}_{\mu_0}(b) = S^{a,\nu}_{\mu_0}(b) = T_{\mu_0 \otimes \nu_0}(a,b) = \infty$. Hence $b \notin uSR^{a,\nu_0}_{\mu_0}$. ∎