

Schnorr randomness for noncomputable measures

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Why Schnorr randomness
is just as good as Martin-Löf randomness
and possibly even better!

Algorithmic randomness

■ Algorithmic randomness

- A branch of computability theory.
- Studies “random points” which behave randomly ...
... with respect to computable statistical tests.
- Is a computable version of Solovay’s random forcing.

■ Martin-Löf randomness

- The most well-known randomness notion.
- Some consider it the “correct” randomness notion
(similar to how Church-Turing is “correct” computability notion).

■ Schnorr randomness

- A more effective variant of Martin-Löf randomness.
- Very closely connected to constructive/computable analysis.
- The reason I don’t think Martin-Löf randomness is “correct” notion.

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Main Ideas of this talk

- Each probability measure has its own set of random points.
- Martin-Löf randomness is robustly defined on both computable and noncomputable probability measures.
- Schnorr randomness has not (until now!) been **robustly defined** on noncomputable probability measures.
- In this talk, I will ...
 - 1 Discuss Schnorr randomness relative to noncomputable oracles.
 - 2 Define Schnorr randomness on noncomp. probability measures.
 - 3 Present theorems demonstrating this definition is the correct one.
 - 4 Discuss other possible definitions and why they do not work.

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Schnorr randomness and Martin-Löf randomness

Definition

A **lower semicomputable function** is

$$t = \sup_n f_n$$

for a computable sequence of computable functions (f_n) .

Definition

For a computable probability measure μ on $2^{\mathbb{N}}$,

- 1 $x \in \text{ML}_\mu$ iff $t(x) < \infty$ for all **ML (integrable) tests** t :
 - t is a nonnegative lower semicomputable function
 - $\int t d\mu < 1$.
- 2 $x \in \text{SR}_\mu$ iff $t(x) < \infty$ for all **Schnorr (integrable) tests** t :
 - t is a nonnegative lower semicomputable function
 - $\int t d\mu = 1$.

Good news / Bad news

Paraphrase from Downey and Hirschfeldt's book

- **Good news:** Most results for ML randomness can be extended to Schnorr randomness **if one makes the objects effective enough.**
- **Bad news:**
 - 1 Schnorr randomness has no universal test.
 - 2 Van Lambalgen's thm for SR fails (really badly).

Theorem (van Lambalgen)

$$x \in \text{ML} \wedge y \in \text{ML}^x \Leftrightarrow x \oplus y \in \text{ML}.$$

Observation (Kjos-Hanssen)

There is $x \oplus y \in \text{SR}$ such that $x \equiv_T y!$

Good news / Good news

Paraphrase from Downey and Hirschfeldt's book

- **Good news:** Most results for ML randomness can be extended to Schnorr randomness **if one makes the objects effective enough.**
- **Good news:**
 - 1 Schnorr randomness has no universal test.
Hence we can compute points that behave randomly!
(Schnorr randomness is the randomness notion for RCA_0 .)
 - 2 Van Lambalgen's thm **works fine** for SR (after natural adjustments).
And it is a quite useful theorem!
 - 3 **Schnorr randomness is very important in the computable analysis of measure theory.**

Outline

- 1 Van Lambalgen's Theorem and Schnorr randomness relativized to noncomputable oracles
- 2 Schnorr randomness for noncomputable measures
- 3 Some example theorems
- 4 Other definitions of Schnorr randomness for noncomputable measures

History of Schnorr randomness (SR) and van Lambalgen's Theorem (vLT) $x \in \text{ML} \wedge y \in \text{ML}^x \Leftrightarrow x \oplus y \in \text{ML}$

Merkle - Miller - Nies - Reimann - Stephan, 2006; Yu, 2007; Kjos-Hanssen

"Easy (\Leftarrow) direction" of vLT does not hold for SR (with usual relativization).

"Folklore"

"Hard (\Rightarrow) direction" of vLT "obviously" holds for SR. (No proof!)

Franklin - Stephan, 2010

Introduced "truth-table SR" (to characterize Schnorr-triviality).

Miyabe, 2011

vLT holds for "truth-table SR".

Franklin - Stephan, 2011

"Hard (\Rightarrow) direction" of vLT holds for SR. (Proof given. Not obvious!)

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Miyabe - R., 2013

Renamed "truth-table SR" \mapsto "uniformly relativized SR".

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R., 201x

vLT holds for "uniformly relativized SR" for **noncomputable measures**.

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vLT holds for "uniformly relativized SR" for **noncomputable measures**.

Uniformly relativized Schnorr randomness

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- A **uniform Schnorr test** is a class $\{t^x\}_{x \in 2^{\mathbb{N}}}$ of tests,
 - t^x is nonnegative and lower semicomputable uniformly from $x \in 2^{\mathbb{N}}$
 - $\int t^x d\mu = 1$ for all $x \in 2^{\mathbb{N}}$.
- $y_0 \in \text{SR}_{\mu}^{x_0}$ iff $t^{x_0}(y_0) < \infty$ for all uniform Schnorr tests $\{t^x\}_{x \in 2^{\mathbb{N}}}$.

ML_{μ}^x can be defined the same way. It is equivalent to the usual definition.

Why did we change the name from “Truth-table Schnorr randomness”?

- Not obvious what it means for a **test** to be “truth-table reducible” to x .
- Not obvious how to handle oracles in other spaces, e.g., $\mathbb{N}^{\mathbb{N}}$ or $\text{Prob}(2^{\mathbb{N}})$.
- Different answers to above can lead to different randomness notions.

Van Lambalgen's theorem for Schnorr randomness

Notation reminder

SR_μ^x will mean Schnorr random **uniformly relative** to the oracle x .

Theorem (Miyabe, Miyabe - R.)

Let λ be fair coin measure. Then

$$x \in \text{SR}_\lambda \wedge y \in \text{SR}_\lambda^x \iff x \oplus y \in \text{SR}_\lambda.$$

Proof of hard (\Rightarrow) direction.

Uses ideas from computable analysis (similar to layerwise computability). \square

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Randomness for noncomputable measures

Definition (Levin)

- A **uniform ML test** is a class $\{t_\mu\}_{\mu \in \text{Prob}(2^{\mathbb{N}})}$ such that
 - t_μ is lower semicomputable uniformly from μ
 - $\int t_\mu d\mu < 1$ for all μ .
- $x_0 \in \text{ML}_{\mu_0}$ iff $t_{\mu_0}(x_0) < \infty$ for all uniform ML tests $\{t_\mu\}_{\mu \in \text{Prob}(2^{\mathbb{N}})}$.

Definition (R.)

- A **uniform Schnorr test** is a class $\{t_\mu\}_{\mu \in \text{Prob}(2^{\mathbb{N}})}$ such that
 - t_μ is lower semicomputable uniformly from μ
 - $\int t_\mu d\mu = 1$ for all μ .
- $x_0 \in \text{SR}_{\mu_0}$ iff $t_{\mu_0}(x_0) < \infty$ for all uniform Schnorr tests $\{t_\mu\}_{\mu \in \text{Prob}(2^{\mathbb{N}})}$.

Randomness for noncomputable oracles and measures

Let X and Y be computable metric spaces (e.g. \mathbb{R} , $C([0,1])$, L^1 , $\text{Prob}(2^{\mathbb{N}})$).

Definition (R.)

- A **uniform Schnorr test** is a class $\{t_{\mu}^a\}_{\mu \in \text{Prob}(X), a \in Y}$ such that
 - t_{μ}^a is lower semicomputable uniformly from μ and a
 - $\int t_{\mu}^a d\mu = 1$ for all μ and a .
- $x_0 \in \text{SR}_{\mu_0}^{a_0}$ iff $t_{\mu_0}^{a_0}(x_0) < \infty$ for all uniform Schnorr tests $\{t_{\mu}^a\}_{\mu \in \text{Prob}(X), a \in Y}$.

Notation: SR_{μ}^a

- The oracle a is in the exponent.
- Measure μ is in the subscript.
- The relativization is always uniform unless stated otherwise.

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Van Lambalgen for noncomputable measures

Let μ and ν be (noncomputable) measures.

Theorem (Folklore?)

$$x \in \text{ML}_{\mu}^{\nu} \wedge y \in \text{ML}_{\nu}^{x, \mu} \Leftrightarrow (x, y) \in \text{ML}_{\mu \otimes \nu}.$$

Theorem (R.)

$$x \in \text{SR}_{\mu}^{\nu} \wedge y \in \text{SR}_{\nu}^{x, \mu} \Leftrightarrow (x, y) \in \text{SR}_{\mu \otimes \nu}.$$

Van Lambalgen for kernels

Let μ be a (noncomputable) measure.

Let $\kappa : 2^{\mathbb{N}} \rightarrow \text{Prob}(2^{\mathbb{N}})$ be a (noncomputable) kernel.

Define

$$(\mu * \kappa)(A) = \int \left(\int \mathbf{1}_A(x, y) d\kappa_x(y) \right) d\mu(x).$$

Theorem (Levin, Takahashi, Bienvenu - Hoyrup - Shen)

$$x \in \text{ML}_{\mu}^{\kappa} \wedge y \in \text{ML}_{\kappa(x)}^{x, \mu, \kappa} \Leftrightarrow (x, y) \in \text{ML}_{\mu * \kappa}^{\kappa}.$$

Theorem (R.)

$$x \in \text{SR}_{\mu}^{\kappa} \wedge y \in \text{SR}_{\kappa(x)}^{x, \mu, \kappa} \Leftrightarrow (x, y) \in \text{SR}_{\mu * \kappa}^{\kappa}.$$

Blind (Hippocratic, oracle-free) randomness for a biased coin

Let μ_p be a p -Bernoulli measure (for a noncomputable $p \in [0, 1]$).

Theorem (Kjos-Hanssen)

$x \in \text{ML}_{\mu_p}$ iff $t(x) < \infty$ for all **blind tests** t ,

- t is a lower semicomputable function (with no knowledge of p or μ_p)
- $\int t d\mu_p < 1$.

Theorem (R.)

$x \in \text{SR}_{\mu_p}$ iff $t(x) < \infty$ for all **blind tests** t ,

- t is a lower semicomputable function (with no knowledge of p or μ_p)
- $q \mapsto \int t d\mu_q$ is computable.

Blind randomness for Π_1^0 classes of measures

Let \mathcal{K} be a Π_1^0 class of measures. (Definition of blind rand. extends to \mathcal{K} .)

Theorem (Bienvenu - Gács - Hoyrup - Rojas - Shen)

The following are equivalent:

- $x \in \text{ML}_\mu$ for some $\mu \in \mathcal{K}$,
- $x \in \text{ML}_\nu$ (blind) for some $\nu \in \mathcal{K}$.

Question

Are the following equivalent?

- $x \in \text{SR}_\mu$ for some $\mu \in \mathcal{K}$,
- $x \in \text{SR}_\nu$ (blind) for some $\nu \in \mathcal{K}$.

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Uniform Schnorr set tests do not exist

We could try to define Schnorr randomness on μ using uniform “set tests”.

Definition

A **uniform Schnorr set test** is a sequence $\{(U_n^\mu)\}_{\mu \in \text{Prob}(2^{\mathbb{N}})}$ where

- U_n^μ is Σ_1^0 uniformly in μ and n ,
- $\mu(U_n^\mu)$ is computable uniformly in μ and n , and
- $\mu(U_n^\mu) \rightarrow 0$.

Theorem (Hoyrup)

The only uniform Schnorr set test is the trivial one:

$$U_n^\mu = \emptyset \quad (\text{for all } n \text{ and } \mu)!$$

Schnorr's approach

Definition (Schnorr)

x is SR_μ (Schnorr) if there is no test of this form:

- a computable measure ν and
- a computable order function f such that

$$\exists^\infty n \quad \frac{\nu(x \upharpoonright n)}{\mu(x \upharpoonright n)} \geq f(n).$$

Observation

This is similar to the blind (Hippocratic) martingale tests used by Kjos-Hanssen - Tavenaux - Thapen for computable randomness.

My guess is that

$$\text{SR}_\mu(\text{Schnorr}) \supsetneq \text{SR}_\mu(\text{uniform}).$$

Reimann - Slaman approach

Definition (Reimann - Slaman)

x is ML_μ (non-uniform / Reimann-Slaman) if there is some name $a \in 2^{\mathbb{N}}$ of μ such that $t(x) < \infty$ for all ML tests (non-uniformly) computable from a .

Theorem (Day - Miller)

ML_μ (uniform / Levin) = ML_μ (non-uniform / Reimann-Slaman).

However, for Schnorr randomness...

Theorem (R.)

SR_μ (uniform) $\not\equiv$ SR_μ (non-uniform). (Too strong!)

Proof idea.

Use that van Lambalgen's theorem fails for non-uniformly relativized SR. \square

Question

Definition

$x \in \text{SR}_{\mu_0}$ (**semi-uniform**) iff

for all **semi-uniform Schnorr tests** $\{t_a\}_{a \in 2^{\mathbb{N}}}$ as below,

$t_{a_0}(x) < \infty$ for some name a_0 of μ_0 ,

- t_a is a lower semicomputable function uniformly in a ,
- $\int t_a d\mu_a = 1$ where μ_a is the measure with name a .

Question

Is SR_{μ} (semi-uniform) equivalent to SR_{μ} (uniform)?

Summary

- Schnorr rand. can be relativized to noncomputable oracles and measures.
- Many theorems for ML randomness hold for Schnorr randomness.
- These results highlight the importance of **uniform computations**.
- I believe these results will be useful in computable analysis.
(Indeed, I already have results!)
- Schnorr randomness is just as good as ML randomness!
...and possibly even better?

Thank You!

These slides will be available on my webpage:

<http://www.personal.psu.edu/jmr71/>

Or just Google™ my name, “Jason Rute”.