Schnorr randomness and computable analysis

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Main Idea

It is well-known that

Martin-Löf randomness is closely connected to computable analysis.

It is not-so-well-known that

Schnorr randomness is even more closely connected to computable analysis.

Outline

1. Computable analysis, a.e. convergence, and Schnorr randomness
2. Why is Schnorr randomness so closely connected to computable analysis?
3. Future directions

Computable analysis

Computable analysis takes a classical theorem in analysis and gives it a computable interpretation.

Theorem (Extreme value theorem)

Given a compact space $\mathbb{X}$ *and a continuous function* $f : \mathbb{X} \to \mathbb{R}$*,*
then $\max_{x \in \mathbb{X}} f(x)$ *exists.*

Computable version

Given an effectively compact space $\mathbb{X}$ and a computable function $f : \mathbb{X} \to \mathbb{R}$, then $\max_{x \in \mathbb{X}} f(x)$ is computable.

Computable version (relativized)

There is a computable algorithm which
takes in a compact space $\mathbb{X}$ and a continuous function $f : \mathbb{X} \to \mathbb{R}$ and
returns $\max_{x \in \mathbb{X}} f(x)$.

Almost-everywhere convergence

Theorem (Lebesgue Differentiation Theorem)

Given an L^1 function $f\colon [0,1] \to \mathbb{R}$, then

$$\frac{1}{2r}\int_{x-r}^{x+r} f(t)\,dt \to f(x) \quad \text{as } r \to 0 \quad \text{for a.e. } x.$$

Computable LDT 1 (Pathak)

If f is L^1-computable and x is Martin-Löf random, then

$$\frac{1}{2r}\int_{x-r}^{x+r} f(t)\,dt \quad \text{converges.}$$

Question (Pathak)

Does the LDT characterize ML randomness? That is

$$x \text{ is Martin-Löf} \Leftrightarrow \frac{1}{2r}\int_{x-r}^{x+r} f(t)\,dt \text{ converges for all } L^1\text{-comp. } f\ ?$$

Answer (R.; Pathak/Rojas/Simpson)

No. The LDT characterizes *Schnorr randomness*.

Effective a.e. convergence

Theorem (Lebesgue Differentiation Theorem)

Given an L^1 function $f\colon [0,1] \to \mathbb{R}$, then

$$\tfrac{1}{2r}\int_{x-r}^{x+r} f(t)\,dt \to f(x) \quad \text{as } r \to 0 \quad \text{for a.e. } x.$$

Computable LDT 2

If f is L^1-computable then

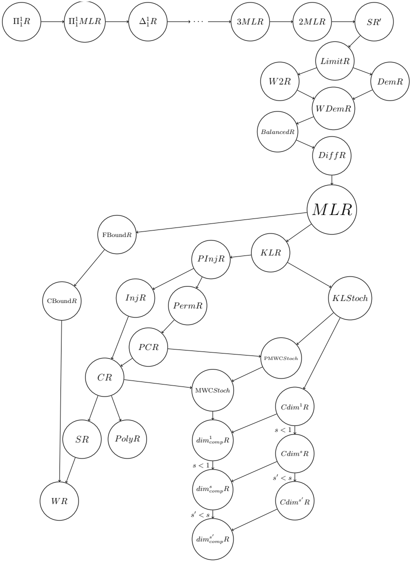
$$\frac{1}{2r}\int_{x-r}^{x+r} f(t)\,dt \to f(x)$$

with an effective rate of almost-everywhere convergence:
There exists a computable rate $\delta(\varepsilon)$ such that

$$\mu\left\{x \,\middle|\, \exists r < \delta(\varepsilon)\ \left|\tfrac{1}{2r}\int_{x-r}^{x+r} f(t)\,dt - f(x)\right| \geqslant \varepsilon\right\} < \varepsilon.$$

Aside: The randomness zoo

- There are many randomness notions.
- Most start out on $2^{\mathbb{N}}$ (coin flipping).
- Some of the more natural ones are:
  - 2-randomness
  - Weak 2-randomness
  - Difference randomness
  - **Martin-Löf randomness**
  - **Computable randomness**
  - **Schnorr randomness**
  - Kurtz randomness
- The natural ones have connections with computable analysis.
- The natural ones can be extended to other computable probability spaces.



Randomness Zoo (Antoine Taveneaux)

Aside: Martin-Löf and Schnorr randomness

Definition

- A **Martin-Löf test** is a computable sequence (U_n) of effectively open sets (uniform sequence of Σ^0_1 sets) such that $\mu(U_n) \leqslant 2^{-n}$.
- A **Schnorr test** is a Martin-Löf test, where $\mu(U_n)$ is uniformly computable.
- x is **Martin-Löf/Schnorr random** (for the measure μ) if $x \notin \bigcap_n U_n$ for each ML/Schnorr tests.

Aside: L^1-computable functions

Definition

An equivalence class $f \in L^1$ is **L^1-computable** if there is a computable sequence of rational polynomials p_n such that $\|f - p_n\|_{L^1} \leqslant 2^{-n}$.

Since f is an *equivalence class*, the value of, say, $f(1)$ is not well-defined.

Definition (Pathak)

Given an L^1-computable f as above, define.

$$\widetilde{f}(x) = \lim_n p_n(x).$$

Proposition (Pathak/Rojas/Simpson; also see R.)

$\widetilde{f}(x)$ is unique and well-defined on Schnorr randoms.

Effective A.E. vs. convergence on Schnorr randomness

Theorem (R. 201x ; Hoyrup/Rojas unpublished)

Assume

1. (f_n) *is a computable sequence of* L^1*-computable functions*
2. $f_n \to f$ *effectively a.e., and*
3. x *is Schnorr random.*

Then $\widetilde{f}_n(x) \to \widetilde{f}(x)$ *converges on Schnorr randoms.*[1]

Remark 1

This result is optimal. For every x not Schnorr random, there is a fast converging sequence of polynomials p_n such that $p_n(x)$ diverges.

Remark 2

To prove convergence on Schnorr randoms, just prove effective a.e. convergence.

[1] Here $\widetilde{f}$ is the canonical representative of f from the previous slide.

Computable Lebesgue Differentiation Thm

Theorem (Computable LDT 3)

If f is L^1-computable and x is Schnorr random, then

$$\frac{1}{2r}\int_{x-r}^{x+r} f(t)\,dt \to \widetilde{f}(x).$$

- Pathak/Rojas/Simpson have a direct proof using quantifier elimination.
- R. has a proof using martingale theory and the convergence theorem on the previous slide.

Absolutely Continuous Measures

Definition

A measure μ is absolutely continuous with respect to Lebesgue measure if there is some density function $f \in L^1$ such that

$$\mu(A) = \int_A f(t)\,dt.$$

Corollary

If μ is absolutely continuous as above with an L^1-computable f, and x is Schnorr random, then

$$\frac{\mu((x-r,x+r))}{2r} \to \widetilde{f}(x) \quad \text{as } r \to 0.$$

Measures and Signed Measures

Theorem (R. unpublished; cf. Brattka/Miller/Nies)

*If μ is a **computable measure** and x is **computably random**, then*

$$\frac{\mu((x-r,x+r))}{2r} \quad \textit{converges.}$$

A signed measure is like a measure, except it takes positive and negative values.

Theorem (R. unpublished; cf. Brattka/Miller/Nies)

*If μ is a **computable signed measure** and x is **Martin-Löf random**, then*

$$\frac{\mu((x-r,x+r))}{2r} \quad \textit{converges.}$$

The above two results characterize computable randomness and Martin-Löf randomness.

Signed Measures and Schnorr randomness

Theorem (R. 201x)

If μ is a computable signed measure, $\|\mu\|_{TV}$ is computable, the Radon-Nykodym derivative $\frac{d\mu}{d\lambda}$ is L^1-computable, x is Schnorr random, then

$$\frac{\mu((x-r,x+r))}{2r} \to \widetilde{\frac{d\mu}{d\lambda}}(x).$$

The proof breaks the measure into an absolutely continuous part and a singular part, and then handle each part individually.

This result characterizes Schnorr randomness.

A.E. convergence results characterizing Schnorr rand.

- Differentiability of bounded variation functions
  - Variation norm and derivative are computable (R.)
  - Function is singular with computable variation norm (R.)
  - Effectively absolutely continuous functions (R., F/K-H/N/S)
  - Comp. Lipschitz functions with comp. variation norm (F/K-H/N/S)
- Doob's Martingale convergence theorem (R.)
  - Uniformly integrable martingales with L^1-computable limit.
  - Limit is 0 and L^1-norm is computable
  - Limit is L^1-computable, L^1-norm is computable
- Ergodic Theorem (Gács/Hoyrup/Rojas and others)
  - System is ergodic or has an effective ergodic decomposition
- Backwards martingale convergence theorem (R.)
  - Limit is L^1-computable
- Monotone convergence theorem (related to Miyabe's work)
  - Maximum L^1-norm is computable

Outline

Constructive mathematics

Imagine you are a "constructivist" (hold that only computable objects exist).
Then Schnorr and Martin-Löf random points do not exist!
Only Schnorr tests and Martin-Löf tests exist.

Definition

Define an **constructive a.e. property** as one where

- (Option A) there is a Schnorr test covering all the exceptions.
- (Option B) there is a ML test covering all the exceptions.

Trivial Thoerem

If property P holds for almost every x, then P holds for some x.

- This fact is constructively valid when using a Schnorr tests.
  - Diagonalize over Schnorr test to find a point satisfying P.
- It is not constructively valid when using Martin-Löf tests.

Schnorr randomness and relativization

Definition (Franklin/Stephan)

Say that x is **Schnorr random uniformly relativize to y** (formally known as **y-truth-table Schnorr random**) if x is not covered by any Schnorr test uniformly computable from y.

Van Lambalgen's theorem for Schnorr randomness (Miyabe; Miyabe/R. 2013)

(x, y) is Schnorr random $\Leftrightarrow$

x is Schorr random and

y is Schnorr random uniformly relative to x.

Remark

VL theorem does not hold for SR using the usual relativization.

Remark (R. in preperation)

Similar ideas can be used to define SR on noncomputable measures.

Outline

Future direction: Understand relativization better

Effective Lebesgue differentiation theorem (R.; Pathak/Rojas/Simpson)

If f is L^1-computable and x is Schnorr random, then

$$\frac{1}{2r}\int_{x-r}^{x+r} f(t)\,dt \quad \text{converges} \quad \text{as } r \to 0.$$

Exercise (What is the relativized version?)

If f is L^1, and x is ____________, then

$$\int_{x-r}^{x+r} f(t)\,dt \quad \text{converges} \quad \text{as } r \to 0.$$

Check all that apply:

1. ☐ Schnorr random.
2. ☒ Schnorr random relativized to f.
3. ☒ Schnorr random uniformly relativized to some code for f.
4. ☐?? Schnorr random uniformly relativized to f.

Future direction: Investigate the following using SR

Only looked at using Martin-Löf randomness

- Effective dimension (only 1 paper using Schnorr randomness)
- Effective capacity (effective potential theory)
- Randomness for classes of measures
- Random closed sets
- Brownian motion
- Geometric measure theory and rectifiable sets

Not yet looked at using randomness

- Continuous-time ergodic theory
- Stochastic calculus and stochastic differential equations
- Calculus of variations and Sobolev spaces
- Harmonic analysis
- Others?

Thank You!

These slides (and the corresponding papers) are available on my webpage:

http://www.personal.psu.edu/jmr71/

Or just Google™ me, “Jason Rute”.