

The computability of martingale convergence

Jason Rute

Carnegie Mellon University
2008 – 2012

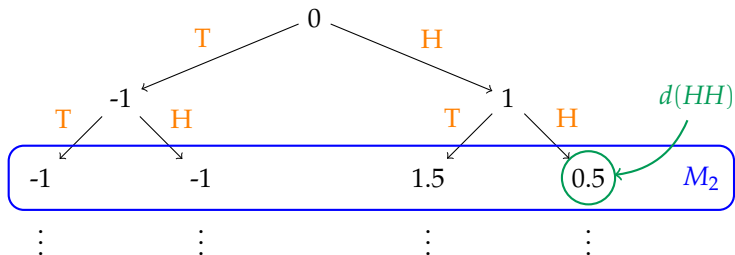
University of Hawaii–Manoa
Spring 2013

Joint Mathematics Meetings
ASL Contributed Paper Session
January 2013

What is a martingale?

Informally: A martingale is a **gambling strategy**.

Example: Bet on a fair coin.



For computability theorists: If $d(\sigma)$ is a martingale in the computability theory sense and $x \in 2^{\mathbb{N}}$, then we are considering

$$M_n(x) = d(x \upharpoonright n).$$

What is a martingale ... really?

Formally:

A martingale is a sequence of integrable functions $M_n: (\Omega, P) \rightarrow \mathbb{R}$ such that

$$\mathbb{E}[M_{n+1} \mid M_0, M_1, \dots, M_n] = M_n.$$

Slogan:

The expectation of the future is the present (conditioned on the past & present).

What are martingales good for?

Probability theory and measure theory

Law of large numbers, de Finetti's theorem, Pólya's urn,
Radon–Nikodym theorem, Lebesgue differentiation theorem

Applications to analysis, finance, computer science, and discrete math

Pricing stocks, Brownian motion, solving differential equations,
Markov processes

Analogous to ... (but martingales are easier to work with)

Ergodic averages, derivatives, harmonic functions

Doob's martingale convergence theorem

Theorem (Doob)

Assume (M_n) is a martingale and

$$\sup_n \|M_n\|_{L^1} < \infty \quad (\text{i.e. } M_n \text{ is } L^1\text{-bounded})$$

Then (M_n) converges pointwise a.e. to an integrable function.

Note: If (M_n) is nonnegative, then

$$\|M_0\|_{L^1} = \|M_1\|_{L^1} = \|M_2\|_{L^1} = \dots$$

Questions

- 1 Is the limit computable (in any sense)? (Comp. analysis)
- 2 Is the convergence effective? (Comp. analysis)
- 3 On what values does the martingale converge?
(Alg. randomness)

Simplifying assumptions

- We WILL assume...
 - We are working in the fair-coin probability measure on $2^{\mathbb{N}}$
...but these results hold in any computable probability space.
 - Each M_n is a computable function (on $2^{\mathbb{N}}$) uniformly in n
...but this can be extended to L^1 -computable functions.
- You MAY assume...
 - All martingales bet on coin flips
...but this is not quite as interesting.
- We WILL NOT assume...
 - M_n is nonnegative
...because taking on debt gives the martingales more power.

Martingale convergence is not effective

The limit of a martingale is **not computable in general**.

Code in the halting problem...

Bet 2^{-e} money that the $2^e 3^s$ th coin is 1 iff the e th Turing machine halts at stage s .

Follow-up Questions

- 1 Under what circumstances is martingale convergence effective?
(Comp. analysis)
- 2 When it is not effective, is there other computable witnesses to the convergence? (Comp. analysis)
- 3 On what values does an effective martingale converge?
(Alg. randomness)

Algorithmic Randomness

Main Idea: A point is **random** if it passes all tests (avoids all probability zero events).

Problem: No points are random! (They all fail some test.)

Solution: Consider “computable tests”. (There are countably many tests, so there is a measure one set of “algorithmic randoms”.)

Note to set theorists: This is basically an effective version of Solovay’s random forcing.

Three randomness notions

Schnorr randomness

- Very robust
- Most computable/constructive randomness notion
- Closely related to computable analysis, reverse mathematics

Computable randomness (recursive randomness)

- Seems ad hoc, but is more robust than one might think
- Has some nice properties, as well as some odd properties
- Is related to computable analysis (more than one might think)

Martin-Löf randomness

- Very robust
- Best behaved randomness notion.
- Closely related to computable analysis, reverse mathematics

Three randomness notions and their definitions

Definition

A point $a \in 2^{\mathbb{N}}$ is ... random for all lower semicomputable functions $t: 2^{\mathbb{N}} \rightarrow [0, \infty]$ such that ... we have $t(a) < \infty$.

- **Schnorr random** (characterization due to Miyabe):

$$\int t(x) dx = 1$$

- **computable random** (characterization due to R.):

$$\forall \sigma \int_{[\sigma]} t(x) dx < \mu([\sigma])$$

for some computable measure μ .

- **Martin-Löf random** (characterization due to Levin?):

$$\int t(x) dx < 1$$

Martingale convergence and randomness

	Condition on limit	Condition on bound
Schnorr	– M_∞ is L^1 -computable	$\sup \ M_n\ _{L^2}$ is computable $\sup \ M_n\ _{L^1}$ is computable
Computable	\mathcal{F}_∞ is computable	$\sup \ M_n\ _{L^1}$ is computable
Martin-Löf	– –	$\sup \ M_n\ _{L^2}$ is finite $\sup \ M_n\ _{L^1}$ is finite

Martingale convergence and Schnorr randomness

	Condition on limit	Condition on bound
Schnorr	– M_∞ is L^1 -computable	$\sup \ M_n\ _{L^2}$ is computable $\sup \ M_n\ _{L^1}$ is computable
Computable	\mathcal{F}_∞ is computable	$\sup \ M_n\ _{L^1}$ is computable
Martin-Löf	– –	$\sup \ M_n\ _{L^2}$ is finite $\sup \ M_n\ _{L^1}$ is finite

Schnorr randomness

Theorem (R.)

The following are equivalent for $x \in 2^{\mathbb{N}}$.

- x is Schnorr random.
- $M_n(x)$ converges for all effective martingales (M_n) such that
 - $\sup_n \|M_n\|_{L^1}$ is computable,
 - $M_\infty := \lim_n M_n$ is L^1 -computable.
- $M_n(x)$ converges for all effective martingales (M_n) such that
 - $\sup_n \|M_n\|_{L^2}$ is computable.
- $M_n(x)$ converges for all effective martingales (M_n) such that
 - (M_n) bets on coin flips (computability theoretic martingale),
 - (M_n) is nonnegative,
 - $M_\infty := \lim_n M_n$ is L^1 -computable.

Convergence and randomness

Definition

We say that (f_n) converges **effectively a.u. (a.e.)** if there is a computable $n(\varepsilon, \delta)$ such that

$$(\forall \varepsilon > 0, \delta > 0) \mu(\{x \mid (\forall m > n(\varepsilon, \delta)) |f_n(x) - f_m(x)| > \varepsilon\}) < \delta$$

“the rate of pointwise convergence is computable for all x outside of an arbitrarily small set.”

Lemma (Hoyrup, Rojas; R.; implicit in Pathak, Rojas, Simpson)

Take a computable sequence (f_n) of computable functions.

If f_n converges effectively a.u., then $f_n(x)$ converges on Schnorr randoms x .

Martingales and convergence

Theorem (R.)

Assume (M_n) is an effective martingale such that

- $\sup_n \|M_n\|_{L^1}$ is computable,
- $M_\infty := \lim_n M_n$ is computable.

Then (M_n) converges effectively a.u.

Hence, $M_n(x)$ converges on Schnorr randoms.

If (M_n) is L^2 -bounded, then M_∞ and $\sup_n \|M_n\|_{L^1}$ are both computable from the L^2 -bound.

Proof Sketch.

Use the limit to find a convergence subsequence.

Use the L^1 -bound to control the deviation from this subsequence.

(Actual proof decomposes the martingale into cases first.)

Martingale convergence and Martin-Löf randomness

	Condition on limit	Condition on bound
Schnorr	– M_∞ is L^1 -computable	$\sup \ M_n\ _{L^2}$ is computable $\sup \ M_n\ _{L^1}$ is computable
Computable	\mathcal{F}_∞ is computable	$\sup \ M_n\ _{L^1}$ is computable
Martin-Löf	– –	$\sup \ M_n\ _{L^2}$ is finite $\sup \ M_n\ _{L^1}$ is finite

Martin-Löf randomness

Theorem (Takahashi; Dean; Merkle-Mihalovic-Slaman; R.)

The following are equivalent for $x \in 2^{\mathbb{N}}$.

- x is Martin-Löf random.
- $M_n(x)$ converges for all effective martingales (M_n) such that
 - $\sup_n \|M_n\|_{L^1} < \infty$.
- $M_n(x)$ converges for all effective martingales (M_n) such that
 - $\sup_n \|M_n\|_{L^2} < \infty$.
- $M_n(x)$ converges for all effective martingales (M_n) such that
 - (M_n) bets on coin flips (computability theoretic martingale),
 - $\sup_n \|M_n\|_{L^1} < \infty$ (do not assume M_n is nonnegative).

Non-effective martingale convergence

Even if the convergence is not effective, there are ways to describe the convergence in an effective manner.

Doob: If mart. (M_n) is L^1 -bounded, then the expected number of upcrossings is bounded, as well as the expected supremum/infimum.

$$\mathbb{E}[\text{upcrossings}(a,b)] \leq \frac{a + \sup_n \|M_n\|_{L^1}}{b-a} \quad \text{and} \quad \mathbb{E}[\sup_n |M_n|] < \infty$$

Lépingle: If mart. (M_n) is L^2 -bounded, then the expected square-variation is bounded.

$$\mathbb{E} \left[\sup_{n_0 < \dots < n_j} \sum_{k=0}^{j-1} |M_{n_{k+1}}(x) - M_{n_k}(x)|^2 \right] \leq \left(\|M_0\|_{L^2} + \sup_n \|M_n\|_{L^2} \right)^2$$

Both of these results quickly show convergence on Martin-Löf randoms. (Takahashi; Dean; R.; others)

Martingale convergence and computable randomness

	Condition on limit	Condition on bound
Schnorr	– M_∞ is L^1 -computable	$\sup \ M_n\ _{L^2}$ is computable $\sup \ M_n\ _{L^1}$ is computable
Computable	\mathcal{F}_∞ is computable	$\sup \ M_n\ _{L^1}$ is computable
Martin-Löf	– –	$\sup \ M_n\ _{L^2}$ is finite $\sup \ M_n\ _{L^1}$ is finite

Computable randomness

Say that a sub- σ -algebra $\mathcal{F} \subseteq \text{Borel}$ is **computable** if $\mathbb{E}[- \mid \mathcal{F}]$ is a computable operator on L^1 .

Theorem

The following are equivalent for $x \in 2^{\mathbb{N}}$.

- x is computably random.
- (R.) $M_n(x)$ converges for all effective martingales (M_n) such that
 - $\mathcal{F}_\infty := \sigma(M_0, M_1, \dots)$ is computable,
 - $\sup_n \|M_n\|_{L^1}$ is computable.
- (Folklore) $M_n(x)$ converges for all effective martingales (M_n) s.t.
 - (M_n) bets on coin flips (computability theoretic martingale),
 - (M_n) is nonnegative.

Main idea

The σ -algebra $\mathcal{F}_\infty = \sigma(M_0, M_1, \dots)$ is the information learned during the gambling process. Knowing that \mathcal{F}_∞ is computable means knowing the information learned **and not learned** while gambling.

The proof is graduate-level probability theory. It relies heavily on the ability to compute $\mathbb{E}[- \mid \mathcal{F}_\infty]$.

Indeed, being able to compute conditional expectation has a lot of applications to algorithmic randomness. For example...

Corollary (R.)

If $T: 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ is a computable measure-preserving map such that $\sigma(T)$ is computable, then T preserves computable randomness.

Extensions and Related Work

Other types of martingales

- Reverse martingales (R.—same basic results)
- Continuous time martingales (current project)
- Martingales on directed sets (future project?)

Martingale convergence is analogous to other convergence results:

Pointwise and Mean Ergodic Theorems

Analogous results by Avigad, Franklin, Gács, Gerhardy, Hoyrup, Rojas, Simic, Towsner, and V'yugen (and others).

Lebesgue Differentiation Thm and Lebesgue's Thm

Analogous results by Brattka, Freer, Kjos-Hanssen, Miller, Nies, Pathak, Rojas, R., Simpson, Stephan (and others).

Applications

Effective Theorems (whose proofs may use martingales)

- Law of Large Numbers
 - Schnorr
 - Gács, Hoyrup, Rojas using ergodic theorem
 - R. using martingales
- de Finetti's Theorem
 - Freer and Roy
 - Hoyrup using ergodic decomposition
 - R. using martingales
- Radon-Nikodym Theorem
 - Hoyrup, Rojas, Weihrauch using von Neumann's pf. and martingales
 - R. using martingales
- Rademacher's Thm in multiple dimensions (future project?)
- Dirichlet boundary problems (future project?)

Thank You!

These slides will be available on my webpage:

math.cmu.edu/~jrute

Or just Google™ me, “Jason Rute”.